

Problem Set I

Content: Sampling Distributions, CLT and Order Statistics

Question 1

A particular brand of dishwasher soap is sold in three sizes: 25 oz, 40 oz, and 65 oz. Twenty percent of all purchasers select a 25-oz box, 50% select a 40-oz box, and the remaining 30% choose a 65-oz box. Let X_1 and X_2 denote the package sizes selected by two independent purchasers.

- Determine the sampling distribution of $\bar{X}$, calculate $E(\bar{X})$, and compare to μ .
- Determine the sampling distribution of the sample variance S^2 , calculate $E(S^2)$, and compare to σ^2 .

Question 2

Let $Y_1, Y_2, \dots, Y_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2 .

- Then, for a positive constant k , consider:

$Y = \frac{1}{2kn} \sum_{i=1}^n Y_i$ Show that Y is normally distributed with mean $\mu_Y = \frac{\mu}{2k}$ and variance $\sigma_{Y}^2 = \frac{\sigma^2}{4k^2n}$.

- Consider $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$. Show that $\bar{Y}$ is normally distributed with mean $\mu_{\bar{Y}} = \mu$ and variance $\sigma_{\bar{Y}}^2 = \frac{\sigma^2}{n}$.

Question 3

Let $X_1, \dots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Find $E(s^2)$, $V(s^2)$.

Question 4

Let Z_1, Z_2 be a random sample of size 2 from $N(0,1)$ and X_1, X_2 a random sample of size 2 from $N(1,1)$. Suppose the Z_i 's are independent of X_j 's. Find the sampling distribution of

- $\bar{X} + \bar{Z}$
- $(Z_1 + Z_2) / \sqrt{[(X_2 - X_1)^2 + (Z_2 - Z_1)^2] / 2}$

- c. $\left[(X_1 - X_2)^2 + (Z_1 - Z_2)^2 + (Z_1 + Z_2)^2 \right] / 2$
- d. $(X_2 + X_1 - 2)^2 / (X_2 - X_1)^2$

Question 5

When a production process is operating correctly, the resistance in ohms of components produced has a normal distribution with standard deviation 3.6. A random sample of four components was taken. What is the probability that the sample variance is bigger than 30?

Question 6

A student in Turkey is studying for University Entrance Exam. She/he has created a contingent plan of solving a certain number of questions every day. For him/her to keep up with the plan, she/he needs to solve 100 questions in less than 2 hours every night. The times it takes for him/her solve a question are independent random variables with mean 1.5 minutes and variance 1.0. Claim with statistical reference that it is highly unlikely for the student to keep up with the plan.

Question 7

A pollster believes that 20% of the voters in a certain area favor a bond issue. If 64 voters are randomly sampled from the large number of voters in this area, approximate the probability that the sampled fraction of voters favoring the bond issue will not differ from the true fraction by more than 0.06.

Question 8

The fracture strength of tempered glass averages 14 (measured in thousands of pounds per square inch) and has standard deviation 2.

- What is the probability that the average fracture strength of 100 randomly selected pieces of this glass exceeds 14.5?
- Find an interval that includes, with probability 0.95, the average fracture strength of 100 randomly selected pieces of this glass.

Question 9

Show that if T has a t distribution with v degrees of freedom, then $U = T^2$ has an F distribution with 1 numerator degree of freedom and v denominator degrees of freedom. ($T = Z/\sqrt{W/v}$, $F = \frac{W_1/v_1}{W_2/v_2}$. Let Z be

a standard normal random variable and let W , W_1 and W_2 a chi-square distributed random variables with v , v_1 and v_2 degrees of freedom respectively. Z and W , W_1 and W_2 are independent)

Question 10

A manufacturer of canned peas is concerned that the mean weight of product be close to the advertised weight. In addition, he does not want too much variability in the weights of the cans of peas; otherwise, a large proportion will differ markedly from the advertised weight. Assume that the population distribution of weights is normal. If a random sample of twenty cans is checked, find the numbers K_1 and K_2 such that

Question 11

Suppose that $n=20$ observations are to be taken on toxicology test measurement which has a normal distribution and that $\sigma^2 = 1, 4$. Let S^2 denote the sample variance of the 20 measurements.

- Find a number a such that $P(a \leq S^2) = 0.975$
- Find a number b such that $P(S^2 \leq b) = 0.975$
- If a and b are as in parts (a) and (b), what is $P(a \leq S^2 \leq b)$?

Question 12

Briefly state the Central Limit Theorem and show that

$$\frac{1}{2n} \int_0^n e^{-x/n} dx = \frac{1}{4}$$

by using the Central Limit Theorem.

Question 13

Suppose that the length of time Y it takes a worker to complete a certain task has the probability density function given by

$$f(y) = \begin{cases} e^{-(y-\theta)}, & y > \theta, 0, \\ elsewhere, \end{cases}$$

where θ is a positive constant that represents the minimum time until task completion. Let $Y_1, Y_2, \dots, Y_n$ denote a random sample of size n from this distribution. Find:

- The density function of

$$Y_{(1)} = \min(Y_1, Y_2, \dots, Y_n).$$

- $E(Y_{(1)})$.