

Problem Set III

Content: Efficiency, Consistency, Sufficiency

Question 1

Let $Y_1, Y_2, \dots, Y_n$ denote a random sample from a population with mean μ and variance σ^2 . Consider the following three estimators for μ :

$$\hat{\mu}_1 = \frac{1}{2}(Y_1 + Y_2), \quad \hat{\mu}_2 = \frac{1}{4}Y_1 + \frac{Y_2 + \dots + Y_{n-1}}{2(n-2)} + \frac{1}{4}Y_n, \quad \hat{\mu}_3 = \bar{Y}$$

- Show that each of the three estimators is unbiased.
- Find the efficiency of $\hat{\mu}_3$ relative to $\hat{\mu}_1$ and $\hat{\mu}_2$, respectively.

Question 2

Let $Y_1, Y_2, \dots, Y_n$ denote a random sample from the uniform distribution on the interval $(0, \theta)$. Two unbiased estimators for θ are

$$\hat{\theta}_1 = \left(\frac{n+1}{n}\right) Y_{(n)} \quad \text{and} \quad \hat{\theta}_2 = 2\bar{Y} \quad \text{where,}$$

$Y_{(n)} = \max(Y_1, Y_2, \dots, Y_n)$. Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ and prove that both estimators are consistent.

Question 3

Let $X_1, X_2, \dots, X_n$ be a random sample from the Poisson distribution with parameter $\lambda > 0$.

- Find the Cramer Rao Lower Bound for λ .
- Is $\bar{X}$ is efficient estimator?

Question 4

Let $X_1, X_2, \dots, X_n$ be a random sample in which X_i possess the probability density function

$$f(x_i | \theta) = \begin{cases} \left(\frac{1}{\theta}\right) e^{-x_i/\theta}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $\theta \geq 0$ for $i = 1, 2, \dots, n$.

- Find a sufficient statistics for θ .
- Based on the sufficient statistics in (a), find an unbiased estimator for θ .
- Find the unique UMVUE estimator for θ
- Show that unique UMVUE estimator you found in part (c) is at the CRLB.

Question 5

Let $X_1, X_2, \dots, X_n$ be a random sample from a population with the following probability density function

$$f(x; \theta) = \begin{cases} ce^{-(x-\theta)} & , \quad x \geq \theta \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

- Calculate the constant c .
- Find a sufficient estimator for θ (call T_n).
- Based on the sufficient T_n , find an unbiased estimator for θ .
- Is the unbiased estimator you found in (c) consistent for θ .
- Find the unique UMVUE estimator for θ .

Question 6

Let $Y_1, Y_2, \dots, Y_n$ be a random sample of size n from a distribution with the probability density function

$$f(y | \theta) = \begin{cases} \left(\frac{2y}{\theta}\right) e^{-y^2/\theta}, & 0 \leq y < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Find a sufficient statistics for θ .

Question 7

Let $X_1, X_2, \dots, X_n$ be an iid sample from the probability distribution $f(x; \theta)$ where

$$f(x; \theta) = \begin{cases} 1/\theta, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}$$

Find the UMVUE estimator for θ .