

Let Y be a discrete random variable with prob. function $P(y)$ & c be a constant. Then $E(c) = c$

Proof:

$$\begin{aligned} E(c) &= \sum_y c \cdot P(y) \\ &= c \cdot \left(\sum P(y) \right) = 1 \text{ since its equal to total probability} \\ &= E(c) = c \end{aligned}$$

Let Y be a discrete random variable with probability function $P(y)$, $g(y)$ be a function of Y , c be a constant

$$E[c \cdot g(y)] = c \cdot E[g(y)]$$

proof:

$$\begin{aligned} E[c \cdot f(y)] &= \sum_y c \cdot g(y) \cdot P(y) \\ &= c \sum_y g(y) \cdot P(y) = c \cdot E[g(y)] \end{aligned}$$

Let Y be a discrete random variable with probability function $p(y)$ & $g_1(y), g_2(y) \dots g_k(y)$ be k functions of Y
Then

$$E[g_1(y) + g_2(y) + \dots + g_k(y)] =$$

$$E[g_1(y)] + E[g_2(y)] + \dots + E[g_k(y)]$$

proof:

$$k=2$$

$$E[g_1(y) + g_2(y)] = \sum_y [g_1(y) + g_2(y)] P_y$$

$$= \sum_y g_1(y) \cdot P(y) + \sum_y g_2(y) P(y)$$

$$= E[g_1(y)] + E[g_2(y)]$$

Let Y be discrete random variable with prob. func. $P(y)$, then $V(y) \sigma^2 = E[(Y - \mu)^2]$

$$= E(Y^2) - \mu^2$$

Proof:

$$\begin{aligned} \sigma^2 &= E[(Y - \mu)^2] = E(Y^2 - 2\mu Y + \mu^2) \\ &= E(Y^2) - E(2\mu Y) + E(\mu^2) \end{aligned}$$

since μ is constant, applying previous theorems

$$\sigma^2 = E(Y^2) - 2\mu \underbrace{E(Y)}_{\mu} + \mu^2 \quad \text{therefore}$$

$$= E(Y^2) - 2\mu^2 + \mu^2 = E(Y^2) - \mu^2$$

Ex: calculate variance of Y , representing # of points rolled with a die

$$\mu = E(Y) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6} = 7/2$$

$$E(Y^2) = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + \dots + 36 \cdot \frac{1}{6} = 91/6$$

$$\sigma^2 = E(Y^2) - \mu^2$$

$$\frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Bernoulli Distribution
 Bernoulli experiment $\begin{cases} \text{success} \\ \text{failure} \end{cases}$

prob. distribution $\rightarrow P(Y) = p^y (1-p)^{1-y}$

Binomial Prob. Distribution

Repeated trials play an important role in prob & stats when # of trials is fixed

A binomial experiment possesses following properties

- ① Experiment consists of a fixed number, n identical trials
- ② Each trial has two outcomes $\begin{cases} \text{success} \\ \text{failure} \end{cases}$
- ③ Prob. of success on a single trial is equal to some value p , this remains same for each trial
 Prob. of failure is $= q = (1-p)$

- ④ Trials are independent.
- ⑤ Random variable of interest y , # of successes observed during n , trials

Prob. of getting y successes & $n-y$ failures in a specific order is $p^y \cdot q^{n-y}$

There is one factor p for each success & one factor $q = (1-p)$ for each failure

y factors p & $n-y$ factors q are all multiplied together by virtue of assumption of independence

Since this probability applies to any sequence of n trials in which there are y successes & $(n-y)$ failures we have counted how many sequences of this kind there are and then multiply by $p^y \cdot q^{n-y}$ by that number

of ways in which we can select y trials on which there is to be success is $\binom{n}{y}$ and desired

probability for y success in n trials is

$$\binom{n}{y} p^y \cdot q^{n-y}$$

A random variable Y has a binomial distr. & it is referred as a binomial random variable iff its prob. distribution is given by

$$p(y) = \binom{n}{y} p^y q^{n-y}$$

$$y = 0, 1, 2, \dots, \quad 0 \leq p \leq 1$$

Ex: find prob. of getting 5H & 7T in 12 tosses of a coin

$$\begin{aligned} y &= 5 & p &= 1/2 \\ n &= 12 \\ p(5) &= \binom{12}{5} \left(\frac{1}{2}\right)^5 \left(1 - \frac{1}{2}\right)^{12-5} \\ &= \binom{12}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^7 = 0.19 \end{aligned}$$

To find binomial probs., we use binomial prob. tables

$$b(y, n, p) = b(n-y, n, 1-p)$$

The term binomial experiment derives from the fact that each trial results in one of two possible outcomes & that probs. $p(y), y=0, 1, \dots, n$ are terms of binomial expansion

$$(q+p)^n = \binom{n}{0} q^n + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \dots + \binom{n}{n} p^n$$

You will observe that

$$\binom{n}{0} q^n = p(0) \quad \binom{n}{1} p^1 q^{n-1} = p(1)$$

$$p(y) = \binom{n}{y} p^y q^{n-y}$$

It also follows that $p(y)$ satisfy necessary properties for a prob. func. because $p(y)$ is positive for $y=0, 1, \dots, n$ and since $(q+p)=1$

$$\sum_y p(y) = \sum_{y=0}^n \binom{n}{y} p^y q^{n-y} \\ = (q+p)^n = 1^n = \underline{1}$$

Let Y be a random variable based on n trials & success probabilities p , then

$$\mu = E(Y) = n \cdot p, \quad \sigma^2 = V(Y) = n \cdot p \cdot q$$

proof: $E(Y) = \sum_y y \cdot p(y)$
 $= \sum_{y=0}^n y \cdot \binom{n}{y} \cdot p^y q^{n-y}$

First term in sum is 0, hence

$$E(Y) = \sum_{y=1}^n y \cdot \frac{n!}{(n-y)! \cdot y!} p^y q^{n-y}$$
$$= \sum_{y=1}^n \frac{n!}{(n-y)! \cdot (y-1)!} \cdot p^y \cdot q^{n-y}$$

Factor out np out of each term in sum
& let $z = y-1$

$$E(Y) = np \sum_{y=1}^n \frac{(n-1)!}{(n-y)! (y-1)!} \cdot p^{y-1} q^{n-y}$$
$$= np \sum_{z=0}^{n-1} \frac{(n-1)!}{(n-1-z)! z!} \cdot p^z q^{n-1-z}$$
$$= np \sum_{z=0}^{n-1} \binom{n-1}{z} p^z q^{n-1-z}$$

notice that $p(z) = \binom{n-1}{z} p^z q^{n-1-z}$

is binomial prob. function based $n-1$ trials

$$\sum_z p(z) = 1 \Leftrightarrow \text{is it equal to this}$$

$$\mu = E(Y) = n \cdot p$$

Consider a random variable $X = \frac{Y}{n}$, where Y is a random variable having a binomial distribution. This random variable X is proportion of successes in n trials

If Y has a binomial distribution and $X = Y/n$, then $E(X) = p$, $G_X^2 = \frac{p(1-p)}{n}$

Geometric Prob. Distribution

The main difference of geometric prob. distr. from binomial distribution is that instead of number of successes in n trials, geometric random variable Y is # of trial on which first success occurs.

Thus, experiment consists of a series of independent trials that concludes with first success.

A random variable Y has a geometric prob. distr. & it is referred as geometric random variable iff its prob. distr. is given by $P(Y) = q^{y-1} p$, $y = 1, 2, 3, \dots$
 $0 \leq p \leq 1$

Ex: If prob. is 0.75 that an applicant for a driver's licence will pass road test on any given try what is the prob. that he/she will finally pass the test on the 4th try?

$$\begin{aligned}P(4) &= 0.75(1-0.75)^{4-1} \\&= 0.75 \cdot (0.25)^3 \\&= 0.0117\end{aligned}$$

$$y=4$$

$$p=0.75 \quad q=0.25$$

If Y is a random variable with geometric distribution

$$\mu = E(Y) = \frac{1}{p} \quad , \quad \sigma^2 = V(Y) = \frac{1-p}{p^2}$$

Thus, mean value of Y is inversely proportional to p . Since larger values of p (smaller values of q) lead to higher probabilities for smaller values of Y and hence lower probabilities for larger values of Y

The Negative Binomial Distribution

It has a close resemblance with geometric probability distr. Now we are interested in number of trial on which 2nd, 3rd, 4th ... success occurs

instead of # of trial on which first success occurs

A random variable Y has a negative binomial distribution & referred as a negative binomial random variable iff its prob. distr. is

$$p(y) = \binom{y-1}{r-1} p^r q^{y-r}$$

$$y = r, r+1, r+2, \dots \quad 0 \leq p \leq 1$$

Ex:

If prob. is 0.40 that a person exposed to covid-19 will catch it. What is the prob that 10th person exposed to disease will be 3rd to catch it

$$y=10, r=3, p=0.4 \quad q=0.6$$

$$p(y) = \binom{9}{2} (0.4)^3 (0.6)^7$$

$$= 0.0645$$

Mean & Variance of a negative binomial distribution are

$$\mu = E(Y) = \frac{r}{p}, \quad \sigma^2 = V(Y) = \frac{r(1-p)}{p^2}$$

Hypergeometric Probability Distribution

A random variable Y has a hypergeom. distr & it is referred as hypergeometric random variable iff its prob distr. is given by

$$P(Y) = \frac{\binom{r}{y} \binom{N-r}{n-y}}{\binom{N}{n}}$$

where $y = 1, 2, \dots, n$

$$y \leq r, \quad r-y \leq N-r$$

For sampling without replacement number of successes in n trials is a random variable having a hypergeometric distribution with parameters n, N, r

Ex: As part of an air pollution survey an inspector decides to examine exhaust of 6 of companies 24 trucks. If 4 of company's trucks emit excessive amounts of pollutants to air, what is the prob. that none of them will be included in the inspectors sample

$$y=0, r=4, N=24, n=6$$

$$p(0) = \frac{\binom{4}{0} \binom{20}{6}}{\binom{24}{6}} = 0.288$$

Mean & Variance of a Hypergeometric distribution are

$$\mu = E(Y) = \frac{n \cdot r}{N} \quad \sigma^2 = V(Y) = n \left(\frac{r}{N} \right) \left(\frac{N-r}{N} \right) \left(\frac{N-n}{N-1} \right)$$

Poisson Distribution

A random variable Y has a poisson distribution & referred as poisson random variable iff it's prob dist. is

$$P(Y) = \frac{\lambda^y}{y!} e^{-\lambda} \quad y=1, 2, \dots, \lambda > 0$$

Poisson distribution provides a good model for probability distribution of # Y of rare events that occur in time

where λ is the average value of Y therefore

Mean & Variance of Poisson distr are

$$\mu = E(Y) = \lambda, \quad \sigma^2 = V(Y) = \lambda$$

Ex: In a factory, 2 of employees are late in a week on average. What is the prob that 3 employees are late in a given selected week?

$$P(3) = \frac{2^3}{3!} e^{-2}$$

$$y=3 \quad \lambda=2$$

$$= \frac{8 \cdot (0.135)}{6} = 0.18$$

Let Y be # of successes resulting from independent trials, each with prob. of success p . Poisson distr. provides a good approximation to binomial probabilities with $n \geq 20$, $p = 0.05$ when $n \geq 100$ & $n \cdot p \geq 10$, approximation is excellent

Prob. func. of approximating distr. is

$$p(y) = \frac{e^{-np} (np)^y}{y!} \quad \text{for } y = 0, 1, 2, \dots$$

Ex: A manufacturer of transistors claims that components are 99.9% reliable, prob that a transistor is defective is 0.01. If you buy 800 transistors, what is the prob. that none is defective

$$y=0 \quad p=0.001 \quad n=800$$

$$\frac{e^{-800 \cdot 1/1000} (800 \cdot 1/1000)^0}{0!} = (0.449329)$$

Moments & Moment Generating Function
(mgf)

The k^{th} moment about origin of a random variable Y , denoted by μ'_k is expected value of Y^k

$$\mu'_k = E(Y^k) = \sum_y y^k \cdot p(y)$$

$$\mu'_1 = E(Y) = \sum_y y \cdot p(y)$$

is called mean of distr. of Y or mean of Y , μ .

The k^{th} moment about mean of a random variable, denoted by μ_k is expected value of $(Y-\mu)^k$ when Y is discrete

$$\mu_k = E[(Y-\mu)^k] = \sum_y (y-\mu)^k p(y)$$

$\mu_0 = 1$, $\mu_1 = 0$ for any random variable

Second moment about mean is called as variance,

μ_2 , denoted by $(\sigma^2, \text{Var}(Y), V(Y))$

$$\mu_2 = E[(Y - \mu)^2] = \sum_y (y - \mu)^2 p(y)$$

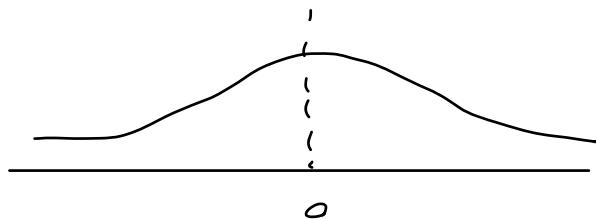
$$\sigma^2 = \underbrace{\mu_2'} - \underbrace{\mu^2} = E(Y^2) - [E(Y)]^2$$

The 3rd moment about mean, denoted by μ_3 , describes symmetry or skewness of a distribution

$\alpha_3 \Rightarrow$ coefficient of skewness

$$\alpha_3 = \frac{E[(Y - \mu)^3]}{\sigma^3} = \frac{\mu_3}{\sigma^3}$$

If $\alpha_3 = 0$ we have a symmetric distribution



$\alpha_4 \Rightarrow$ coefficient of kurtosis

$$\alpha_4 = \frac{E[(Y - \mu)^4]}{\sigma^4} = \frac{\mu_4}{\sigma^4}$$



If Y has a variance σ^2 , then $\text{Var}(aY+b) = a^2\sigma^2$

Although the moments of most distributions can directly be determined by evaluating necessary sums. An alternative procedure generally provides considerable simplifications.

This technique uses moment generating func. mgf for a random variable provides us all moments for a random variable by one simple expression.

The mgf of a random var. Y , where it exists, is given by

$$m(t) = E(e^{tY}) = \sum_y e^{ty} \cdot p(y)$$

mgf for Y exists if there exists a positive constant b such that $m(t)$ is finite for $|t| \leq b$

Independent variable is t , we are interested in values of t in neighborhood of 0

To explain why it is called as mgf substitute for e^{ty} its maclauren's series expansion

$$e^{ty} = 1 + ty + \frac{t^2 y^2}{2!} + \frac{t^3 y^3}{3!} + \dots + \frac{t^k y^k}{k!}$$

$$m(t) = E(e^{ty}) = \sum_y \left[1 + ty + \frac{t^2 y^2}{2!} + \dots + \frac{t^k y^k}{k!} \right] p(y)$$

$$= \sum_y p(y) + t \sum_y y p(y) + \frac{t^2}{2!} \sum_y y^2 p(y) + \dots + \frac{t^k}{k!} \sum_y y^k p(y)$$

$$= 1 + \mu t + \mu_2' \cdot \frac{t^2}{2!} + \dots + \mu_k' \frac{t^k}{k!}$$

If $m(t)$ exists, than for any positive intiger k

$$\left. \frac{d^k m(t)}{dt^k} \right|_{t=0} = m^{(k)}(0) = \mu_k'$$

If a & b are constants, then

$$\textcircled{1} m_{y+a}(t) = E[e^{(y+a)t}] = e^{at} m_y(t)$$

$$\textcircled{2} m_{by}(t) = E[e^{b yt}] = m_y(bt)$$

$$\textcircled{3} m_{\frac{y+a}{b}}(t) = E[e^{(\frac{y+a}{b})t}] = e^{\frac{a}{b}t} m_y\left(\frac{t}{b}\right)$$

Ex: Random Variable Y can assume values 1 & -1 with $1/2$ prob. Find a moment generating function

b) find four moments about origin

$$\begin{aligned} a) E(e^{ty}) &= e^{t(1)} \left(\frac{1}{2} \right) + e^{t(-1)} \left(\frac{1}{2} \right) \\ &= \frac{1}{2} (e^t + e^{-t}) \end{aligned}$$

$$e^t = 1 + \cancel{t} + \frac{t^2}{2!} + \frac{\cancel{t^3}}{3!} + \frac{t^4}{4!} + \dots$$

$$e^{-t} = 1 - \cancel{t} + \frac{t^2}{2!} - \frac{\cancel{t^3}}{3!} + \frac{t^4}{4!} \dots$$

$$\frac{1}{2} (e^t + e^{-t}) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots$$

$$m(t) = 1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \mu_3 \frac{t^3}{3!} + \mu_4 \frac{t^4}{4!} + \dots$$

$$\text{Thus, } \mu_1 = 0, \mu_2 = 1, \mu_3 = 0, \mu_4 = 1$$

the odd moments in this exercise are all zero, even moments are all one

If $m(t)$ exist for a prob. distribution it is unique

If mgf's for two random variables Y & Z are equal then Y & Z must have same prob distr.

Probability-Generating Function (pgf)

Let Y be an integer valued random variable for which $P(Y=i) = p_i$ where $i = 0, 1, 2, \dots$. The probability generating func $P(t)$ for Y is defined to be

$$P(t) = E(t^Y) = p_0 + p_1 t + p_2 t^2 + \dots$$

$$\sum_{i=0}^{\infty} p_i t^i \quad \text{for all values } t \text{ such that}$$

$P(t)$ is finite coefficient of t^i in $P(t)$ is prob. p_i . that's why its called pgf

Repeated differentiation of $P(t)$ yields factorial moments for random varb. Y

The k^{th} factorial moment for random variable Y is defined as

$$\mu_{[k]} = E[Y(Y-1)(Y-2)\dots(Y-k+1)]$$

where k is a positive integer

$$\mu_{[1]} = \mu$$

$\mu_{[2]} = E(Y(Y-1))$ is useful in finding variance of binomial, geometric, poisson random variables

If $P(t)$ is pgf for an integer valued random var. Y , then k^{th} factorial moment is given by

$$\left. \frac{d^k p(t)}{dt^k} \right|_{t=1} = p^{(k)}(1) = \mu_{[k]}$$

Exercises:

Ex: If prob. distribution of x is given by

$$f(x) = \left(\frac{1}{2}\right)^x \text{ for } x=1, 2, 3, \dots$$

Show that $E(2^x)$ does not exist

$$E(2^x) = \sum_{x=1}^{\infty} 2^x \cdot p(y) \rightarrow \sum_{x=1}^{\infty} 2^x \cdot \left(\frac{1}{2}\right)^x$$

$$\sum_{x=1}^{\infty} \left(2 \cdot \frac{1}{2}\right)^x \rightarrow \sum_{x=1}^{\infty} (1)^x = 1 + 1 + 1 \dots \equiv \infty$$

So $E(2^x)$ does not exist.

Petersburg Paradox

Ex 2: Find expected value of a discrete random variable x having prob. distr.

$$f(x) = \frac{|x-2|}{7} \text{ for } x = -1, 0, 1, 3$$

$$\sum y \cdot p(y) \rightarrow \sum x \cdot \frac{1x-2}{7}$$

$$E(x) = (-1) \left(\frac{3}{7} \right) + (0) \left(\frac{2}{7} \right) + 1 \left(\frac{1}{7} \right) + 3 \left(\frac{1}{7} \right)$$

$$= 1/7$$

Ex 3: Prob. that a landlord will sell a piece of property at a profit of \$3000 is $3/20$, Prob. that he/she will sell it at a profit of \$1500 is $7/20$, prob that he/she will sell it at a profit of break-even level (aka no profit) is $7/20$ and prob that he/she will lose \$1500 is $3/20$

What is her expected profit

\$3000	$\rightarrow 3/20$	$E(x) = x \cdot p(x)$ $\frac{3000}{450} \cdot \frac{3}{20} + \frac{1500}{225} \cdot \frac{7}{20} + 0 \cdot \frac{7}{20} + \frac{-1500}{-225} \cdot \frac{3}{20}$ $1500 \left(\frac{7}{20} - \frac{3}{20} \right)$ $\frac{1800}{1800} \cdot \frac{4}{20} =$
\$1500	$\rightarrow 7/20$	
0	$\rightarrow 7/20$	
-1500	$\rightarrow 3/20$	

$$450 + 300 = 750 = \mu$$

Ex 4: In a certain city, incompatibility is given as legal reason in 70% of all divorce cases. Find prob. that 5 of next 6 divorce cases will claim incompatibility

↳ binomial distr.

$$b(x, n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$\begin{array}{l} n=6 \\ x=5 \\ \theta=0.7 \end{array} \quad \binom{6}{5} 0.7^5 (0.3) = 0.3025$$

Ex 5: # of monthly breakdowns in computer systems is a random variable having Poisson distr. with $\lambda = 1.8$ find prob that this system will function for a month

a) without a breakdown

b) with only one breakdown $P(y) = \frac{\lambda^y}{y!} e^{-\lambda}$

a) $\lambda = 1.8$, $x=0$

$$P(0, 1.8) = \frac{(1.8)^0 (e^{-1.8})}{0!} = 0.1653$$

$$b) P(1, 1.8) = \frac{(1.8)^1 (e^{-1.8})}{1!} = 0.2975$$

Ex 6: If events A, B, C are independent, show that

- a) A and $B \cap C$ are independent
- b) A and $B \cup C$ are independent

Ex 7: For any event A , show that A & $\bar{A}$ are independent