

## Multivariable Transformations

Suppose we are given joint density/distr. of two random variables  $x_1$  &  $x_2$

→ we want to determine prob. distr./density of r.v.  
 $y = U(x_1, x_2)$

### discrete case

If the relationship b/w.  $y$  &  $x_1$  with  $x_2$  held constant or the relationship b/w.  $y$  &  $x_2$  with  $x_1$  held constant permits  
→ we can find joint distr. of  $y$  &  $x_2$  or  $x_1$  &  $y$ , then sum on values of other r.v.s to get marginal distr. of  $y$

### Continuous Case

We use a transformation formula

$$g(y, x_2) = f(x_1, x_2) \left| \frac{\partial x_1}{\partial y} \right| \rightarrow \text{Then we integrate out other variable to get marginal density of } y$$

$$g(y, x_1) = f(x_1, x_2) \left| \frac{\partial x_2}{\partial y} \right|$$

Let  $f(x_1, x_2)$  be value of joint prob. density of cont. r.v.  $x_1$  &  $x_2$  at  $(x_1, x_2)$

If funcs. given by  $y_1 = u_1(x_1, x_2)$   $y_2 = u_2(x_1, x_2)$   
if they are partially differentiable with respect to  $x_1$  &  $x_2$  & represent a one-to-one transformation for all values within range of  $x_1$  &  $x_2$  for which  $f(x_1, x_2) \neq 0$   
then for these values of  $x_1$  &  $x_2$ , equations  
 $y_1 = u_1(x_1, x_2)$  &  $y_2 = u_2(x_1, x_2)$  can uniquely be solved to give

$x_1 = w_1(y_1, y_2)$  &  $x_2 = w_2(y_1, y_2)$  and for corresponding values of  $y_1$  &  $y_2$  joint probability of  $y_1 = u_1(x_1, x_2)$  &  $y_2 = u_2(x_1, x_2)$  is given by

$$g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)] |J|$$

$J \Rightarrow$  Jacobian transformation

$$J = \begin{vmatrix} \partial x_1 / \partial y_1 & \partial x_1 / \partial y_2 \\ \partial x_2 / \partial y_1 & \partial x_2 / \partial y_2 \end{vmatrix}$$

If we have  $n$  variables Jacobian is

$$J = \begin{vmatrix} \partial x_1 / \partial y_1 & \partial x_1 / \partial y_2 & \dots & \partial x_1 / \partial y_n \\ \partial x_2 / \partial y_1 & \partial x_2 / \partial y_2 & \dots & \partial x_2 / \partial y_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial x_n / \partial y_1 & \partial x_n / \partial y_2 & \dots & \partial x_n / \partial y_n \end{vmatrix}$$

Ex:

If joint pdf of  $x_1, x_2$  &  $x_3$  is given by  

$$f(x_1, x_2, x_3) = \begin{cases} e^{-(x_1 + x_2 + x_3)} & \text{for } x_1 > 0, x_2 > 0, x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the joint density of  $y_1 = x_1 + x_2 + x_3$   
 $y_2 = x_2$   
 $y_3 = x_3$

$$x_1 = y_1 - y_2 - y_3$$

$$x_2 = y_2$$

$$x_3 = y_3$$

$$J = \begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$g(y_1, y_2, y_3) = \begin{cases} e^{-y_1} & y_2 > 0, y_3 > 0, y_1 > y_2 + y_3 \\ 0 & \text{elsewhere} \end{cases}$$

Ex:

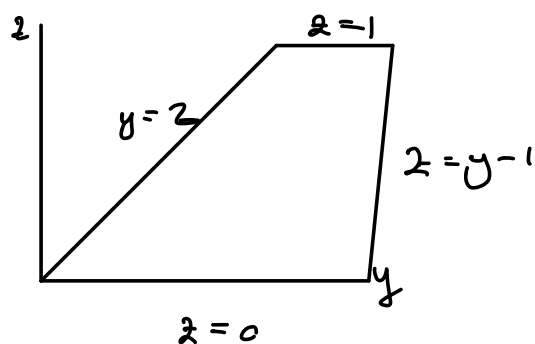
$$f(x_1, x_2) = \begin{cases} 1 & \text{for } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find joint density of  $Y = x_1 + x_2$  &  $Z = x_2$   
 $x_1 = y - z$        $x_2 = z$

$$J = \begin{vmatrix} \partial x_1 / \partial y & \partial x_1 / \partial z \\ \partial x_2 / \partial y & \partial x_2 / \partial z \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$f(y, z) = 1 \quad ||| = 1$$

because this transformation is one to one mapping region  $0 < x_1 < 1$  &  $0 < x_2 < 1$  in  $x_1 x_2$  plane and region  $z < y < z+1$ ,  $0 < z < 1$  in  $yz$  plane



$$g(y, z) = \begin{cases} 1 & \text{for } z < y < z+1 \text{ & } 0 < z < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Ex: let  $x_1, x_2$  be two continuous r.v.s having joint pdf

$$f(x_1, x_2) = \begin{cases} 4x_1 x_2 & \text{for } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find joint pdf of  $Y_1 = X_1^2$   $Y_2 = X_1 X_2$

$$\begin{aligned} x_1 &= \sqrt{y_1} & \partial x_1 / \partial y_1 &= \frac{1}{2\sqrt{y_1}} & \partial x_2 / \partial y_1 &= -\frac{1}{2} y_2 y_1^{-3/2} \\ x_2 &= \frac{y_2}{\sqrt{y_1}} & \partial x_1 / \partial y_2 &= 0 & \frac{\partial x_2}{\partial y_2} &= \frac{1}{\sqrt{y_1}} \end{aligned}$$

$$J = \begin{vmatrix} \frac{1}{2\sqrt{y_1}} & 0 \\ -\frac{1}{2} y_2 y_1^{-3/2} & \frac{1}{\sqrt{y_1}} \end{vmatrix} = \frac{1}{2y_1}$$

$$g(y_1, y_2) = \begin{cases} 4\sqrt{y_1} \cdot \frac{y_2}{\sqrt{y_1}} \cdot \frac{1}{2y_1} = \frac{2y_2}{y_1} & \text{over region bounded by } y_1=1, y_2=0, y_1=y_2^2 \\ 0 & \text{elsewhere} \end{cases}$$

Ex: Let  $x$  &  $y$  be two cont. r.v.s having joint pdf

$$f(x, y) = \begin{cases} 24xy & , \quad 0 < x < 1, \quad 0 < y < 1, \quad x+y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find joint pdf of  $Z = x+y$  &  $W = x$

$$x = w \quad y = z - w$$

$$\partial x / \partial w = 1, \quad \partial x / \partial z = 0, \quad \partial y / \partial w = -1, \quad \partial y / \partial z = 1$$

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 \quad g(w, z) = \begin{cases} 24 \cdot w \cdot (z - w) & \text{over the region bounded by } w=0, z=1, z=w \\ 0 & \text{elsewhere} \end{cases}$$

prob. discrete  
continuous

- 1 - ilk üç chapterden 1 soru
- 2 - multivariate random var. korelasyon katsayısı hesaplama
  - ↳ corr. coeff  $\rightarrow$  bir tane olursa bulun
  - ↳ conditional expectation
- 3 - Transformations — multivariable  $\rightarrow$  bir tane  
— ? diğer üç yöntemden biri

gamma vb. olursa formül var