

Functions of Random Variables

→ Given a set of r.v. $Y_1, Y_2, Y_3, \dots, Y_n$ & their joint prob. distr. / density of same r.v.s
 $U(Y_1, Y_2, \dots, Y_n)$

For finding prob. distr. of U

- method of distr. func.
- method of transformations
 - one var
 - several var
- method of moment generating func.

Method of Distribution Functions

→ Method of obtaining prob density of a func. of continuous r.v.s consists of

- ① Finding its distribution function
- ② Its prob density by differentiation

If $Y_1, Y_2, \dots, Y_n$ are continuous r.v.s with a given joint pdf, prob density of $U(Y_1, Y_2, \dots, Y_n)$ is obtained by first determining an expression for

$$F_U(u) = P(U \leq u) = P[U(Y_1, Y_2, \dots, Y_n) \leq u]$$

then by differentiating

$$f_U(u) = \frac{dF_U(u)}{du}$$

Ex:

If prob. density of x is

$$f(x) = \begin{cases} 6x(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Prob density of $Y = X^3$

Let $G(y)$ denote value of distr. func of Y at y

$$G(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3})$$

$$= \int_0^{y^{1/3}} 6x(1-x) dx = \int_0^{y^{1/3}} 6x - 6x^2 = \left. \frac{6x^2}{2} - \frac{6x^3}{3} \right|_0^{y^{1/3}}$$

$$= 3y^{2/3} - 2y$$

$$g(y) = \frac{dG(y)}{dy}$$

$$g(y) = \begin{cases} 2(y^{1/3} - 1) & \text{for } 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$g(y) = 2(y^{1/3} - 1)$$

Ex: If joint pdf of x_1 & x_2 is

$$f(x_1, x_2) = \begin{cases} 6e^{-3x_1 - 2x_2} & x_1 > 0, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find pdf of $Y = x_1 + x_2$

$$F(y) = \int_0^y \int_0^{y-x_2} 6e^{-3x_1 - 2x_2} dx_1 dx_2$$

$$= 1 + 2e^{-3y} - 3e^{-2y}$$

$$f(y) = \frac{dF(y)}{dy} \quad f(y) = 6(e^{-2y} - e^{-3y})$$

$$f(y) = \begin{cases} 6(e^{-2y} - e^{-3y}) & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Ex: If pdf of x is

$$f(x) = \begin{cases} 2xe^{-x^2}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find pdf of $Y = x^2$

$$G(y) = P(Y \leq y) = P(x^2 \leq y) = P(x \leq \sqrt{y}) = \int_0^{\sqrt{y}} 2xe^{-x^2} dx$$

$$\text{let } u = x^2$$

$$du = 2x dx$$

$$G(y) = \int_0^y e^{-u} du = -e^{-u} \Big|_0^y = 1 - e^{-y}$$

$$g(y) = \frac{dG(y)}{dy} = e^{-y}$$

$$f(y) = \begin{cases} e^{-y}, & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Ex: If joint density of x & y is

$$f(x, y) = \begin{cases} e^{-(x+y)} & , x > 0 \quad y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find pdf of $z = \frac{x+y}{2}$ $x+y = 2z$
 $y = 2z - x$

$$G(z) = P\left(z \leq \frac{x+y}{2}\right)$$

$$= \int_0^{2z} \int_0^{2z-x} e^{-x} e^{-y} \cdot dy dx = \int_0^{2z} e^{-x} [-e^{-y}] \Big|_0^{2z-x} dx =$$

$$\int_0^{2z} e^{-x} [1 - e^{-x-2z}] dx = \int_0^{2z} (e^{-x} - e^{-2z}) dz$$

$$= [-e^{-x} \times e^{-2z}] \Big|_0^{2z} = -e^{-2z} - 2ze^{-2z} + 1 \quad \frac{dG(z)}{dz} =$$

$$2e^{-2z} - 2e^{-2z} + 4ze^{-2z} = 4ze^{-2z}$$

$$f(z) = \begin{cases} 4ze^{-2z} & z > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Ex: If x has an exponential distr: with parameter β
use distribution function technique to find pdf of r.v.
 $y = \sqrt{x}$

$$\begin{aligned} G(y) &= P(Y \leq y) = P(\sqrt{x} \leq y) = P(x \leq e^y) \\ &= \int_0^{e^y} \frac{1}{\beta} e^{-x/\beta} \cdot dx = -e^{-x/\beta} \Big|_0^{e^y} = 1 - e^{-(1/\beta)e^y} \end{aligned}$$

$$\frac{dg(y)}{dy} = \frac{1}{\beta} e^y e^{-(1/\beta)e^y} \quad g(y) = \begin{cases} \frac{1}{\beta} e^y e^{-(1/\beta)e^y} & -\infty < y < \infty \end{cases}$$

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Ex:

If $Y = |X|$ show that

$$f(y) = \begin{cases} f(y) + f(-y), & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $f(x)$ is value of prob. density of X at x &

$g(y)$ is value of prob. density of Y at y

Use this result to find prob density of $Y = |X|$ when $|X| \sim N(0,1)$

For $y > 0$, we have

$$G(y) = P(Y \leq y) = P(|X| \leq y) = P(-y \leq X \leq y) = F(y) - F(-y)$$

By differentiation

$$g(y) = f(y) + f(-y)$$

Since $|x|$ cannot be negative, $g(y) = 0$ for $y < 0$

$$g(y) = \begin{cases} f(y) + f(-y) & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

If X has a standard normal distr. $Y = |X|$ it follows that

$$g(y) = n(y; 0,1) + n(y; 0,1) = 2n(y; 0,1)$$

$$g(y) = \begin{cases} 2n(y; 0,1) & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$



Method of Transformations: One Variable

$$U(Y_1, Y_2, \dots, Y_n)$$

We show how prob. distribution/density is obtained without getting its distr. func.

1- Discrete case

In discrete case, no real prob. as long as relationship bw. values of x & $Y(x)$ is one-to-one

Ex:

If Y is number of heads obtained in four tosses of a coin, find prob. distr. of $x = \frac{1}{1+Y}$

Using formula for binomial distribution $n=4$ $p=1/2$ prob. distr of Y is given by

$$P(Y) = \binom{n}{y} p^y \cdot (1-p)^{n-y}$$

Y	0	1	2	3	4
$f(Y)$	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

Using relationship $x = \frac{1}{1+Y}$
substitute values of Y for
values of x Prob distr of x is

x	1	$1/2$	$1/3$	$1/4$	$1/5$
$g(x)$	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

or

make substitution directly into binomial distribution

with $n=4$ & $p=1/2$

$Y = \frac{1}{x} - 1$ is substituted into

$$f(y) = \binom{4}{y} \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{4-y} = \binom{4}{y} \left(\frac{1}{2}\right)^4 \text{ for } y=0,1,2,3,4$$

$$g(x) = f\left(\frac{1}{x} - 1\right) = \binom{4}{\frac{1}{x} - 1} \left(\frac{1}{2}\right)^4 \text{ for } x=1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$$

In this example probabilities remain unchanged, only difference is that in the result they are associated with various values of x instead of corresponding values of y

Transformation (or change of variable) technique in discrete case

is done if r.s. is one to one.

If relationship is not one to one

Ex:

If Y is # of heads obtained in four tosses of a coin, find prob. distr. of Z which is $(Y-2)^2 = Z$

calculating probs. of $h(z)$ associated with various values of z we get

$$h(0) = f(2) = \frac{6}{16} = \frac{3}{8}$$

$$h(1) = f(1) + f(3) = \frac{4}{16} + \frac{4}{16} = \frac{1}{2}$$

$$h(4) = f(0) + f(4) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

$$y=0,1,2,3,4$$

$$y=0 \Rightarrow z=4$$

$$y=1 \Rightarrow z=1$$

$$y=2 \Rightarrow z=0$$

$$y=3 \Rightarrow z=1$$

$$y=4 \Rightarrow z=4$$

z	0	1	4
$h(z)$	$3/8$	$1/2$	$1/8$

Ex: If Y has hypergeometric distr. with $N=6, n=2, r=3$
Find prob. distr. of X , # of successes minus # of failures

$$p(y) = \frac{\binom{r}{y} \binom{N-r}{n-y}}{\binom{N}{n}} \quad p(0) = \frac{\binom{3}{0} \binom{3}{2}}{\binom{6}{2}} = \frac{3}{15} = \frac{1}{5}$$

$$p(1) = \frac{\binom{3}{1} \binom{3}{1}}{\binom{6}{2}} = \frac{9}{15} = \frac{3}{5} \quad p(2) = \frac{\binom{3}{2} \binom{3}{0}}{\binom{6}{2}} = \frac{3}{15} = \frac{1}{5}$$

of successes $\rightarrow Y$

of failures $\rightarrow 2-Y$

$$X = Y - (2-Y)$$

$$Y \quad 0 \quad 1 \quad 2$$

$$X \quad -2 \quad 0 \quad 2$$

X	-2	0	2
$g(x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

Ex: For previous example find prob. distr. of $Z = (Y-1)^2$

$$Y \quad 0 \quad 1 \quad 2$$

$$Z \quad 1 \quad 0 \quad 1$$

Z	0	1
$p(z)$	$3/5$	$2/5$

Ex: If X has a binomial distr with $n=3, p=1/3$
find prob. distr. of a) $Y = \frac{X}{1+X}$

$$b) Z = (X-1)^2$$

$$p(0) = 1 \cdot \frac{8}{27} = \frac{8}{27} \quad p(1) = 3 \cdot \frac{1}{3} \cdot \frac{4}{9} = \frac{12}{27}$$

$$p(2) = 3 \cdot \frac{1}{3} \cdot \frac{2}{9} = \frac{6}{27} \quad p(3) = 1 \cdot \frac{1}{27} = \frac{1}{27}$$

a)	2	0	1	2	3						
	y	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$		y	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$
							p(y)	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

b)	x	0	1	2	3					
	y	1	0	1	16		z	0	1	16
							p(z)	$\frac{12}{27}$	$\frac{14}{27}$	$\frac{1}{27}$

Continuous Case

In order to perform transformation in continuous case assume that

- i- func. given by $y = u(x)$ is differentiable
- ii- either increasing or decreasing for all values within range of x for which $f(x) \neq 0$

Thus, inverse function given by $x = w(y) \Rightarrow$ exist for all corresponding values of y is differentiable except where $u'(x) = 0$

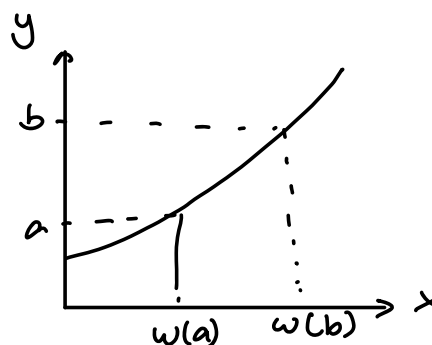
to avoid points where $u'(x)$ might be 2 or 0

$\rightarrow$ we don't include endpoints of interval for which prob. densities are non zero

Let $f(x)$ be value of prob. density of continous r.v. x at x . If func. given by $y = u(x)$ is differentiable either increasing or decreasing for all values within range of x for which $f(x) \neq 0$, then for these values of x , $y = u(x)$ can be uniquely solved for x to find $x = w(y)$ & for corresponding values of y prob. density of $y = u(x)$ is given by $g(y) = f[w(y)] |w'(y)|$ provided $u'(x) \neq 0$ elsewhere $f(y) = 0$

Proof:

(i) let $y = u(x)$ is increasing



x must take on a value $w(a)$ & $w(b)$ when y takes values a & b
hence:

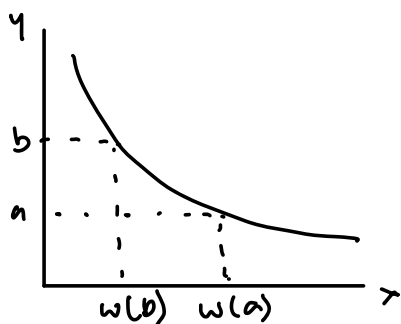
$$P(a < Y < b) = P[w(a) < X < w(b)] = \int_{w(a)}^{w(b)} f(x) \cdot dx = \int_a^b f[w(y)] \cdot w'(y) \cdot dy$$

where we performed change of var. $y = u(x)$ or $x = w(y)$ in interval

Integral gives prob. density of Y if $w'(y)$ exists, we can write

$$g(y) = f[w(y)] \cdot w'(y)$$

(ii) let $y = u(x)$ is decreasing



x must take on a value bw $w(b)$ & $w(a)$ when y takes on value a & b

$$P(a < Y < b) = P[w(b) < X < w(a)] = \int_{w(b)}^{w(a)} f(x) \cdot dx \Rightarrow \int_b^a f[w(y)] w'(y) \cdot dy$$

$$= - \int_a^b f[w(y)] w'(y) dy$$

by change of variable as before

$$g(y) = -f[w(y)] w'(y) \text{ since}$$

$$w'(y) = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \text{ is positive when } y = u(x) \text{ is increasing}$$

= $w'(y)$ is negative when $y = u(x)$ is decreasing

we can combine two cases

$$g(y) = f[w(y)] |w'(y)|$$

Ex: If X has an exponential distr. given by

$$f(x) = \begin{cases} e^{-x} & , x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find pdf of $Y = \sqrt{X}$

Equation $y = \sqrt{x} \Rightarrow$ has a unique inverse $x = y^2$ which yields $w'(y) = \frac{dx}{dy} = 2y$

$$g(y) = f[w(y)] |w'(y)| = e^{-y^2} |2y| = 2ye^{-y^2}, y > 0$$

$$f(y) = \begin{cases} 2ye^{-y^2} & , \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Ex: Pdf of X is

$$f(x) = \begin{cases} \frac{kx^3}{(1+2x)^2} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where k is an appropriate constant, find pdf of c.v. $Y = \frac{2X}{1+2X}$ & calculate k

$$Y = \frac{2X}{1+2X} \Rightarrow Y(1+2X) = 2X \Rightarrow Y + 2XY = 2X \Rightarrow Y = 2X - 2XY$$

$$= 2X(1-Y) \Rightarrow 2X = \frac{Y}{1-Y}$$

$$X = \frac{Y}{2(1-Y)}$$

$$g(y) = \frac{dx}{dy} f(x) \Rightarrow 2 \cdot \frac{dx}{dy} = \frac{(1-y)^{-4}}{(1-y)^2} = \frac{1}{(1-y)^2} \Rightarrow \frac{dx}{dy} = \frac{1}{2(1-y)^2}$$

$$g(y) = \frac{k y^3 (1-y)^6}{8 (1-y)^3} \cdot \frac{1}{2(1-y)^2} = \frac{k}{16} y^3 (1-y) \Rightarrow$$

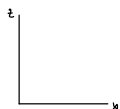
$$\int_0^1 \frac{k}{16} y^3 (1-y) \cdot dy = 1 \Rightarrow k = 320$$

$$g(y) = \begin{cases} 20y^3(1-y) & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

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Ex: If X has a st. normal distr., find prob density of $Z = X^2$

Func. $Z = X^2$ is decreasing for negative values of x & increasing for positive values of x



transformation from x to z can be found in two steps

1- Find prob. density of $y = |x|$

2- Find prob. density of $z = y^2 = x^2$
for $y > 0$

$G(y) = P(y \leq y) = P(|x| \leq y) = P(-y \leq x \leq y) = F(y) - F(-y)$
by differentiation $g(y) = f(y) + f(-y)$

If $x \sim N(0, 1)$ & $y = |x|$

$$g(y) = n(y, 0, 1) + (-y; 0, 1) = 2n(y; 0, 1) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \text{ for } y > 0$$

0 elsewhere

For 2nd step

Func. given by $x = y^2$ is increasing for $y > 0$, that is for all values of y for which $g(y) \neq 0$ & $y = \sqrt{z}$

$$\frac{dy}{dz} = \frac{1}{2} z^{-1/2}$$

$$h(z) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}z} \left| \frac{1}{2} z^{-1/2} \right| = \frac{1}{\sqrt{2\pi}} z^{-1/2} e^{-1/2 z} \text{ for } z > 0$$

0 elsewhere

Method of Moment-Generating Funcs

If $y_1, y_2, \dots, y_n$ are independent r.v.s $U = y_1 + y_2 + \dots + y_n$

$$M_U(t) = \prod_{i=1}^n m_{y_i}(t) \quad \text{where } m_{y_i}(t) \text{ is value mgf of } y_i \text{ at } t$$

Ex: let Y_1, Y_2, Y_3 be independent & identically distributed r.v.s having prob. distr. func.

$$f(y) = \frac{e^{-\lambda} \lambda^y}{y!}, \quad y=0, 1, 2, \dots$$

i.e. $Y_1, Y_2, Y_3 \sim \text{Poisson}(\lambda)$

Find distr. of $U = Y_1 + Y_2 + Y_3$

$$\begin{aligned} M_U(t) &= m_{Y_1+Y_2+Y_3}(t) = E(e^{t(Y_1+Y_2+Y_3)}) = (m_Y(t))^3 \\ &= (e^{\lambda(e^t-1)})^3 = e^{3\lambda(e^t-1)} \end{aligned}$$

It is moment generating function of a poisson r.v. with mean 3λ , $U = Y_1 + Y_2 + Y_3 \sim \text{Poisson}(3\lambda)$

Ex: If $X_1, X_2, \dots, X_n$ are independent r.v.s having exponential distr. with same parameter β

Find prob. density of r.v. $Y = X_1 + X_2 + \dots + X_n$

$$\text{mgf of exp. distr} \Rightarrow m_{X_i}(t) = (1 - \beta t)^{-1}$$

$$m_Y(t) = \prod_{i=1}^n (1 - \beta t)^{-1} = (1 - \beta t)^{-n}$$

Method of mgfs can be used to establish useful results for normally distributed r.v.s

let $Y_1, Y_2, \dots, Y_n$ be independent normally distributed r.v.s with $E(Y_i) = \mu_i$, $V(Y_i) = \sigma_i^2$, $i=1, 2, \dots, n$

let $a_1, a_2, \dots, a_n$ be constants

$$\text{If } U = \sum_{i=1}^n a_i Y_i = a_1 Y_1 + a_2 Y_2 + \dots + a_n Y_n$$

U is normally distr. r.v. with $E(U) = \sum_{i=1}^n a_i \mu_i = a_1 \mu_1 + a_2 \mu_2 + \dots + a_n \mu_n$

$$V(u) = \sum_{i=1}^n a_1 \sigma_i^2 + a_2 \sigma_2^2 - \dots + a_n \sigma_n^2$$

Let $y_1, y_2, \dots, y_n$ be independent normally distr. rvs with $E(y_i) = \mu_i$ & $V(y_i) = \sigma_i^2$, $i=1, 2, \dots, n$

$$z_i = \frac{y_i - \mu_i}{\sigma_i}$$

$$\sum z_i^2 \sim \chi_n^2$$

↳ chi-square distr with n degrees of freedom

If n indep. s.t. normal rvs are squared & added $\Rightarrow$ resulting sum has a χ^2 distr with n d.f