

Hypothesis Testing II

the general model

$$(1) Y_t = \beta_0 + \overbrace{\beta_1 X_{t1} + \dots + \beta_{k_1} X_{tk_1}}^{k_1 \text{ coefficients}} + \underbrace{\beta_{k_1+1} X_{t(k_1+1)} + \dots + \beta_k X_{tk}}_{k_2 \text{ coefficients}} + u_t$$

$$k_1 + k_2 = k \text{ slope coefficients}$$

$$H_0: \beta_{k_1+1} = 0$$

$$\beta_{k_1+2} = 0$$

$$\vdots$$

$$\beta_k = 0$$

under H_0 :

$$(2) Y_t = \beta_0 + \beta_1 X_{t1} + \dots + \beta_k X_{tk} + u_t$$

Restricted & Unrestricted Models

(constrained / unconstrained)

↳ the model obtained under the restriction imposed by H_0 on the unrestricted model

unrestricted model $\Rightarrow (1)$

restricted model $\Rightarrow (2)$

example:

$$(1) Y_t = \beta_0 + \beta_1 X_t + \beta_2 Q_t + \beta_3 W_t + \beta_4 Z_t + U_t$$

$$H_0 : \beta_3 = \beta_4 = 0$$

$$H_A : \beta_3 \neq 0 \\ \beta_4 \neq 0$$

Under H_0

$$(2) Y_t = \beta_0 + \beta_1 X_t + \beta_2 Q_t + U_t$$

testing

Suppose,

We estimate both the unrestricted and restricted models

$$S_R : SSR \text{ from the restricted model} : \sum \hat{U}_{tR}^2$$

$$S_U : \text{"unrestricted"} : \sum \hat{U}_{tU}^2$$

If the restriction has no significant effect on the explanation of Y_t , then S_R will not be much different (smaller) than S_U

$$\Rightarrow S_R - S_U : \text{small}$$

$$F_P = \frac{S_R - S_U}{S_U} \cdot \frac{df_U}{P} \sim F_{P, df_U}^\alpha$$

$$R H_0 \text{ if } F_P > F_{P, df}^\alpha$$

Note that

$$F_p = \frac{S_R - S_u}{S_u} \cdot \frac{df_u}{p} = \frac{\sum \hat{u}_{tR}^2 - \sum \hat{u}_{tU}^2}{\sum \hat{u}_{tU}^2} \cdot \frac{df_u}{p}$$

and

$$R^2 = 1 - \frac{\sum \hat{u}_t^2}{\sum y_t^2}$$

$$\Rightarrow \sum \hat{u}_t^2 = (1 - R^2) \sum y_t^2$$

then

$$\sum \hat{u}_{tU}^2 = (1 - R_u^2) \sum y_t^2$$

$$\sum \hat{u}_{tR}^2 = (1 - R_R^2) \sum y_t^2$$

$$F_p = \frac{(1 - R_R^2) \cancel{\sum y_t^2} - (1 - R_u^2) \cancel{\sum y_t^2}}{(1 - R_u^2) \cancel{\sum y_t^2}} \cdot \frac{df_u}{p}$$

$$\left. \right\} F_p = \frac{R_u^2 - R_R^2}{1 - R_u^2} \cdot \frac{df_u}{p}$$

Similarly:

$$F_p = \frac{R_{Ru}^2 - R_{RR}^2}{1 - R_{Ru}^2} \cdot \frac{df_u}{p}$$

****** if both restricted and un restricted models have an intercept term: R^2

if there is no intercept term in either or both of the restricted and un restricted models: R_r^2

example:

$$(1) Y_t = \beta_0 + \beta_1 X_{t1} + \dots + \beta_k X_{tk} + U_t$$

$$\left. \begin{array}{l} H_0: \beta_1 = 0 \\ \beta_2 = 0 \\ \vdots \\ \beta_k = 0 \end{array} \right\} \begin{array}{l} \text{testing for the joint significance of the} \\ \text{expl. variables.} \\ \\ \text{Testing for the significance of the model} \\ \text{whole} \end{array}$$

$$(2) \text{ Under } H_0: Y_t = \beta_0 + U_t$$

$$F_p = \frac{R_u^2 - R_{u^2}^2}{1 - R_u^2} \cdot \frac{df_u}{p} = 0$$

$$F_p = \frac{R_u^2}{1 - R_u^2} \cdot \frac{df_u}{p}$$

the significance of the model can be determined by using the R^2 of that model

Hypothesis Testing when $p=1$

$$\left. \begin{array}{l} H_0: \beta_i = \beta_i^* \\ H_A: \beta_i \neq \beta_i^* \end{array} \right\} \rightarrow t_c = \frac{\hat{\beta}_i - \beta_i^*}{\hat{\sigma}_{\hat{\beta}_i}} \sim t_{df_u}, \alpha/2$$

$$i) F_1 = (t_c)^2 \Leftrightarrow H_0: \beta_i = 0$$

$$F_1 = \frac{S_R - S_u}{S_u} \cdot \frac{df_u}{1}$$

$$* F_1 = \frac{R_i^2 - R_{-i}^2}{1 - R_i^2} \cdot \frac{df_u}{1} = \frac{R_u^2 - R_{u^2}^2}{1 - R_u^2} \cdot \frac{df_u}{1}$$

↑ if both models have an intercept term

$$* t_{\hat{\beta}_i}^2 = f_i = \frac{r_{y_i}^2}{1 - r_{y_i}^2} \cdot df_u$$

$$\begin{array}{l} H_0: \beta_i = \beta_j \\ H_A: \beta_i \neq \beta_j \end{array} \quad \left. \begin{array}{l} H_0: \beta_i - \beta_j = 0 \\ H_A: \beta_i - \beta_j \neq 0 \end{array} \right\}$$

i) Direct Test

$$t_i = \frac{\hat{\beta}_i - \hat{\beta}_j}{\hat{\sigma}(\hat{\beta}_i - \hat{\beta}_j)} = \frac{\hat{\beta}_i - \hat{\beta}_j}{\sqrt{\text{Var}(\hat{\beta}_i) + \text{Var}(\hat{\beta}_j) - 2 \text{Cov}(\hat{\beta}_i, \hat{\beta}_j)}}$$

ii) Regression

impose the restriction, estimate the restr. model

$$\hookrightarrow F = \frac{S_R - S_u}{S_u} \cdot \frac{df_u}{p}$$

example

$$(1) Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \beta_3 X_{t3} + u_t$$

$$H_0: \beta_1 = \beta_2$$

$$H_A: \beta_1 \neq \beta_2$$

impose restriction $\Rightarrow$

$$\begin{aligned} Y_t &= \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \beta_3 X_{t3} + u_t \\ &= \beta_0 + \beta_1 (\underbrace{X_{t1} + X_{t2}}_{Z_t}) + \beta_3 X_{t3} + u_t \end{aligned}$$

$$(2) Y_t = \beta_0 + \beta_1 Z_t + \beta_3 X_{t3} + u_t$$

$$F \rightarrow \frac{S_R - S_u}{S_u} \cdot \frac{df_u}{p} \rightarrow 1$$

example:

$$(1) - I_t = \beta_0 + \beta_1 R_t + \beta_2 R_t + u_t$$

$$(2) - R_T = \beta_0 + \beta_1 \overbrace{AR_t}^{\text{nominal int. rate}} + u_t$$

where $AR_t = R_t - \pi \rightarrow$ inflation rate

$\hookrightarrow$ real int. rate

How to test?

restriction: often not given

$$\hookrightarrow \text{restriction: } \beta_1 + \beta_2 = 0 \Rightarrow \beta_1 = -\beta_2$$

either use t or F statistics

Previous Exam Question

$$1) Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + u_t$$

$$\begin{array}{l} H_0: \beta_1 + \beta_2 = 1 \\ H_A: \beta_1 + \beta_2 \neq 1 \end{array} \quad \left. \begin{array}{l} H_0: \beta_1 + \beta_2 - 1 = 0 \\ H_{A1}: \beta_1 + \beta_2 - 1 \neq 0 \end{array} \right\} \begin{array}{l} H_0: \delta = 0 \\ H_A: \delta \neq 0 \end{array}$$

i) t -test

ii) Reparametrisation

Define $\beta_1 + \beta_2 - 1 = \delta$

$\beta_1 = \delta - \beta_2 + 1 \rightarrow$ insert & re-arrange

$$(1) Y_t = \beta_0 + \delta X_{t1} - \beta_2 X_{t1} + X_{t1} + \beta_2 X_{t2} + u_t$$

$$(2) Y_t - X_{t1} = \beta_0 + \delta X_{t1} + \beta_2 (X_{t2} - X_{t1}) + u_t$$

a) estimate (2), $\delta = 0 \Rightarrow t$ -test

or

(3) under $H_0 \Rightarrow$

$$Y_t - X_{t1} = \beta_0 + \beta_2 (X_{t2} - X_{t1}) + u_t$$

(1) unrestricted model

(3) restricted model

F-test

Example: exam question

$$(1) Y_t = \beta_0 + \beta_1 X_t + \beta_2 Q_t + \beta_3 W_t + u_t$$

$$(2) Z_t = \gamma_0 + \gamma_1 X_t + \gamma_2 Q_t + \gamma_3 W_t + u_t$$

where $Z_t = Y_t - Q_t$

$\rightarrow$ define and then the validity of restriction to obtain 2

$\rightarrow$ given SSR, $\rightarrow$ compute SSR₂

$\rightarrow$ compare (1) & (2)

$R^2!!$

exam question:

Consider the following estimated regression

$$\hat{C}_t = a_0 + 0.92 Y_t \quad (1)$$

$$\hat{C}_t = b_0 + 0.84 X_t \quad (2)$$

$$\hat{X}_t = c_0 + 0.78 Y_t \quad (3)$$

$$\hat{Y}_t = d_0 + 0.55 X_t \quad (4)$$

calculate the OLS estimates of e_1 and e_2 for following regression

$$C_t = e_0 + e_1 Y_t + e_2 X_t + u_t \quad (b)$$

$$C_t - \bar{C} = \hat{e}_0 + \hat{e}_1 Y_t + \hat{e}_2 X_t - \bar{C}$$

$$C_t - \bar{C} = \hat{e}_0 + \hat{e}_1 Y_t + \hat{e}_2 X_t - \hat{e}_0 - \hat{e}_1 \bar{Y} - \hat{e}_2 \bar{X}$$

$$\underbrace{C_t - \bar{C}}_{C_t} = \hat{e}_1 \underbrace{(Y_t - \bar{Y})}_{Y_t} + \hat{e}_2 \underbrace{(X_t - \bar{X})}_{X_t}$$

$$C_t = \hat{e}_1 Y_t + \hat{e}_2 X_t \quad (7)$$

multiply by Y_t

$$\sum C_t Y_t = \hat{e}_1 \sum Y_t^2 + \hat{e}_2 \sum X_t Y_t$$

$\div \sum Y_t^2$

$$\frac{\sum C_t Y_t}{\sum Y_t^2} = \hat{e}_1 \frac{\sum Y_t^2}{\sum Y_t^2} + \hat{e}_2 \frac{\sum X_t Y_t}{\sum Y_t^2}$$

$\hat{a}_1$
 $\hat{e}_1$
 $\hat{e}_2$
 $\hat{c}_1$

$$\hat{a}_1 = \hat{e}_1 + \hat{e}_2 \hat{c}_1 \quad (8)$$

(7) $\times X_t$

$$\sum C_t X_t = \hat{e}_1 \sum X_t Y_t + \hat{e}_2 \sum X_t^2$$

(7) $\div \sum X_t^2$

$$\frac{\sum C_t X_t}{\sum X_t^2} = \hat{e}_1 \frac{\sum X_t Y_t}{\sum X_t^2} + \hat{e}_2 \frac{\sum X_t^2}{\sum X_t^2}$$

$\hat{b}_1$
 $\hat{d}_1$
 $\hat{e}_2$

$$\hat{b}_1 = \hat{d}_1 + \hat{e}_2 \quad (9)$$

$$a_1^n = e_1 + e_2^n c_1^n$$

$$b_1^n = d_1^n + e_2^n$$

$$\begin{aligned} & \rightarrow (1) \\ & 0.92 = \hat{e}_1 + \hat{e}_2 (0.78) \quad \rightarrow (3) \\ & \rightarrow (2) \\ & (0.84) = \hat{e}_1 (0.55) + \hat{e}_2 \\ & \hat{e}_1 = 0.45 \quad \rightarrow (4) \\ & \hat{e}_2 = 0.59 \end{aligned}$$