

Problem Set 10 - Answers

- 1) Two messages are to be sent. The time (min) necessary to send each message has an exponential distribution with parameter $k = 1$, and the two times are independent of each other. It costs \$2 per minute to send the first message and \$1 per minute to send the second. Obtain the density function of the total cost of sending the two messages. **[Hint: First obtain the cumulative distribution function of the total cost, which involves integrating the joint pdf.]**

SOLUTION

Let X and Y be the transmission times, so the joint pdf of X and Y is
 $f(x, y) = f_X(x) \cdot f_Y(y) = e^{-x}e^{-y} = e^{-(x+y)}$ for $x, y > 0$. Define $T = 2X + Y$ = the total cost to send the two messages. The cdf of T is given by
 $F_T(t) = P(T \leq t) = P(2X + Y \leq t) = P(Y \leq t - 2X)$. For $X > t/2$, this probability is zero (since Y can't be negative). Otherwise, for $X \leq t/2$,
 $P(Y \leq t - 2X) = \int_0^{t/2} \int_0^{t-2x} e^{-(x+y)} dy dx = \dots = 1 - 2e^{-t/2} + e^{-t}$ for $t > 0$. Thus, the pdf of T is
 $f_T(t) = F_T'(t) = e^{-t/2} - e^{-t}$ for $t > 0$.

- 2) A type of elevator has a maximum weight capacity Y_1 , which is normally distributed with mean 5000 pounds and standard deviation 300 pounds. For a certain building equipped with this type of elevator, the elevator's load, Y_2 , is a normally distributed random variable with mean 4000 pounds and standard deviation 400 pounds. For any given time that the elevator is in use, find the probability that it will be overloaded, assuming that Y_1 and Y_2 are independent.

SOLUTION

The probability of interest is $P(Y_2 > Y_1) = P(Y_2 - Y_1 > 0)$.

The distribution of $Y_2 - Y_1$ is normal with

$$\mu = 4000 - 5000 = -1000$$

and

$$\sigma^2 = 400^2 + 300^2 = 250,000.$$

Thus,

$$P(Y_2 - Y_1 > 0) = P\left(Z > \frac{0 - (-1000)}{\sqrt{250,000}}\right) = P(Z > 2).$$

Finally,

$$P(Z > 2) = 0.0228.$$

- 3) A machine produces spherical containers whose radii vary according to the probability density function given by

$$f(r) = \begin{cases} 2r, & 0 \leq r \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the probability density function for the volume of the containers.

(Hint : Volume of a sphere is $V = \frac{4}{3}\pi R^3$)

SOLUTION

The volume of the sphere is $V = \frac{4}{3}\pi R^3$, or $R = \left(\frac{3}{4\pi}V\right)^{1/3}$, so that $\frac{dr}{dv} = \frac{1}{3}\left(\frac{3}{4\pi}\right)^{1/3}v^{-2/3}$. Thus,

$$f_V(v) = \frac{2}{3}\left(\frac{3}{4\pi}\right)^{2/3}v^{-1/3}, \quad 0 \leq v \leq \frac{4}{3}\pi.$$

$$f_V(v) = \begin{cases} \frac{2}{3} * \left(\frac{3}{4\pi}\right)^{2/3}v^{-1/3}, & 0 \leq v \leq \frac{4}{3}\pi \\ 0, & \text{elsewhere} \end{cases}$$

- 4) Suppose that X_1 and X_2 have joint probability density function

$$f(x_1, x_2) = \begin{cases} 2 & \text{for } 0 < x_1 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

Compute the joint probability density function of random variables $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_2$.

SOLUTION

Note that the given transformation is a one-to-one transformation. First, we need to find inverses.

$$\begin{aligned}Y_1 &= \frac{X_1}{X_2} \text{ and } Y_2 = X_2 \\X_2 &= Y_2 \text{ and } X_1 = Y_1 Y_2 \\ \text{Thus, } 0 < x_1 < x_2 < 1 &\text{ is if and only if } 0 < y_1 y_2 < y_2 < 1. \\ \text{So, } 0 < y_2 < 1 \text{ and } 0 < y_1 < 1\end{aligned}$$

Next, construct the Jacobian.

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2$$

Note that $|y_2| = y_2$ and y_2 is non-zero for $0 < y_2 < 1$.

Then,

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 2y_2 & \text{for } 0 < y_1 < 1, 0 < y_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

- 5) Let Y_1 and Y_2 be independent random variables with moment-generating functions $m_{Y_1}(t)$ and $m_{Y_2}(t)$, respectively. If a and b are constants and $U = aY_1 + bY_2$, show that the moment-generating function for U is $m_U(t) = m_{Y_1}(at) * m_{Y_2}(bt)$.

SOLUTION

Let Y_1 and Y_2 have moment-generating functions as given and let $U = aY_1 + bY_2$. The moment-generating function for U is $m_U(t) = E(e^{Ut}) = E(e^{(aY_1 + bY_2)t}) = E(e^{(at)Y_1}) * E(e^{(bt)Y_2}) = m_{Y_1}(at) * m_{Y_2}(bt)$.

- 6) Let Y_1 and Y_2 be independent Poisson random variables with means λ_1 and λ_2 , respectively. Find the probability mass function of Y_1+Y_2 .

SOLUTION

The moment generating functions for Y_1 and Y_2 are, respectively, $m_{Y_1}(t) = e^{\lambda_1(e^t-1)}$, $m_{Y_2}(t) = e^{\lambda_2(e^t-1)}$. Since Y_1 and Y_2 are independent, the moment-generating function for Y_1+Y_2 is $m_{Y_1}(t) * m_{Y_2}(t) = e^{(\lambda_1+\lambda_2)(e^t-1)}$. This is the moment-generating function of a Poisson distribution with mean $\lambda_1+\lambda_2$. Hence the pmf of $Y = Y_1+Y_2$ is

$$p(y) = \frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^y}{y!}, \quad y=0,1,2,\dots$$

- 7) Let Z_1 and Z_2 be independent standard normal random variables and $U_1=Z_1$ and $U_2=Z_1+Z_2$. Derive the joint density of U_1 and U_2 .

SOLUTION

By independence, the joint density of Z_1 and Z_2 is the product of the two marginal densities:

$$f(z_1, z_2) = \frac{1}{2\pi} e^{-(z_1^2+z_2^2)/2}, \quad -\infty < z_1 < \infty, -\infty < z_2 < \infty$$

With $U_1=Z_1$ and $U_2=Z_1+Z_2$, we have that $z_1=u_1$ and $z_2=u_2-u_1$. Thus, the Jacobian is

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

Thus, the joint density of U_1 and U_2 is

$$f(u_1, u_2) = \frac{1}{2\pi} e^{-u_1^2} e^{u_1 u_2} e^{-u_2^2/2}, \quad -\infty < u_1 < \infty, -\infty < u_2 < \infty$$

- 8) Suppose that Y_1 and Y_2 are independent, standard normal random variables. Find the density function of $U = Y_1^2 + Y_2^2$.

SOLUTION

It has been shown that the distribution of both Y_1^2 and Y_2^2 is chi-square with $v = 1$. Thus, both have mgf $m(t) = (1 - 2t)^{-1/2}$, $t < 1/2$.

Since Y_1^2 and Y_2^2 are independent, with $U = Y_1^2 + Y_2^2$,

$$m_U(t) = m(t)m(t) = (1 - 2t)^{-1}.$$

Note that this is the mgf of an exponential random variable with $\beta = 2$, so the density function for U is

$$f_U(u) = \frac{1}{2}e^{-u/2}, \quad u \geq 0,$$

(this is also the chi-square distribution with $v = 2$).

9)

The length of time that a machine operates without failure is denoted by Y_1 and the length of time to repair a failure is denoted by Y_2 . After a repair is made, the machine is assumed to operate like a new machine. Y_1 and Y_2 are independent and each has the density function

$$f(y) = \begin{cases} e^{-y}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability density function for $U = Y_1/(Y_1 + Y_2)$, the proportion of time that the machine is in operation during any one operation–repair cycle.

SOLUTION

The joint distribution of Y_1 and Y_2 is $f(y_1, y_2) = e^{-(y_1+y_2)}$, $y_1 > 0, y_2 > 0$. Let $U_1 = \frac{Y_1}{Y_1+Y_2}$, $U_2 = Y_2$. The inverse transformations are $y_1 = u_1 u_2 / (1 - u_1)$ and $y_2 = u_2$ so the Jacobian of transformation is

$$J = \begin{vmatrix} \frac{u_2}{(1-u_1)^2} & \frac{u_1}{1-u_1} \\ 0 & 1 \end{vmatrix} = \frac{u_2}{(1-u_1)^2}.$$

Thus, the joint distribution of U_1 and U_2 is

$$f(u_1, u_2) = e^{-[u_1 u_2 / (1-u_1) + u_2]} \frac{u_2}{(1-u_1)^2} = e^{-[u_2 / (1-u_1)]} \frac{u_2}{(1-u_1)^2}, \quad 0 \leq u_1 \leq 1, u_2 > 0.$$

Therefore, the marginal distribution for U_1 is

$$f_{U_1}(u_1) = \int_0^{\infty} e^{-[u_2 / (1-u_1)]} \frac{u_2}{(1-u_1)^2} du_2 = 1, \quad 0 \leq u_1 \leq 1.$$

Note that the integrand is a gamma density function with $\alpha = 2$, $\beta = 1 - u_1$.