

Bivariate & Multivariate Prob. Distr.

If y_1 & y_2 are discrete r.v. function given by

$$p(y_1, y_2) = f(y_1 = y_1, y_2 = y_2, \dots) = -\infty < y_1 < \infty \\ -\infty < y_2 < \infty$$

for each pair of values (y_1, y_2) within range of y_1 & y_2 is called joint prob. distr. of y_1 & y_2

A bivariate function serve as joint prob. distr. of a pair of discrete r.v. iff its values $p(y_1, y_2)$ satisfy

$$1 - p(y_1, y_2) \geq 0 \text{ for } \forall y_1, y_2$$

$$2 - \sum_{y_1} \sum_{y_2} p(y_1, y_2) = 1$$

For any r.v's y_1, y_2 , the joint bivariate distr. func. (cumulative distr. $F(y_1, y_2)$) is given by

$$F(y_1, y_2) = P(y_1 \leq y_1, y_2 \leq y_2), -\infty < y_1 < \infty, -\infty < y_2 < \infty$$

$$F(y_1, y_2) = \sum_{t_1=-\infty}^{y_1} \sum_{t_2=-\infty}^{y_2} p(t_1, t_2)$$

All these can be generalized to multivariate case where there are n r.v. values of joint prob. distr. of n discrete r.v's $y_1, y_2, \dots, y_n$ are given by

$$p(y_1, y_2, \dots, y_n) = P(y_1 = y_1, y_2 = y_2, \dots, y_n = y_n)$$

for each n -tuple $(y_1, y_2, \dots, y_n)$ within range of r.v's.

Values of their joint prob. distr. func. are

$$F(y_1, y_2, \dots, y_n) = P(y_1 \leq y_1, y_2 \leq y_2, \dots, y_n \leq y_n) \text{ for } \\ -\infty < y_1 < \infty, -\infty < y_2 < \infty, \dots, -\infty < y_n < \infty$$

Ex:

If joint prob. distr. of three discrete r.v's X, Y, Z is

$$P(X, Y, Z) = \frac{(x+y)z}{63} \text{ for } x=1,2 \quad y=1,2,3 \quad z=1,2$$

find prob $X=2, Y+Z \leq 3$

$$P(X=2, Y+Z \leq 3) = P(2, 1, 1) + P(2, 1, 2) + P(2, 2, 1)$$

$$\frac{(2+1)1}{63} + \frac{(2+1)2}{63} + \frac{(2+2)1}{63} = \frac{13}{63}$$

A bivariate function with values $f(y_1, y_2)$ defined over y_1, y_2 plane is called joint prob. density function of continuous r.v.s

$$P[(y_1, y_2) \in A] = \int_{b_1}^{a_1} \int_{a_2}^{b_2} f(y_1, y_2) \cdot dy_1 \cdot dy_2$$

for any region A in y_1, y_2 plane defined by $a_1 \leq y_1 \leq a_2$
 $b_1 \leq y_2 \leq b_2$
 a bivariate function can serve as a joint prob. density func of a pair of continuous rvs Y_1 & Y_2 if its values satisfy

$$1- f(y_1, y_2) \geq 0, \text{ for all } y_1, y_2 \quad -\infty < y_1 < \infty, -\infty < y_2 < \infty$$

$$2- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 dy_2 = 1$$

Ex:

Given joint pdf.

$$f(y_1, y_2) = \begin{cases} \frac{3}{5} y_1 (y_2 + y_1) & \text{for } 0 < y_1 < 1, 0 < y_2 < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find $P[(y_1, y_2) \in A]$ where A is region $\{(y_1, y_2) | 0 < y_1 < \frac{1}{2};$

$$P(0 < y_1 < \frac{1}{2} < y_2 < 2) = \quad 1 < y_2 < 2\}$$

$$\int_1^2 \int_0^{1/2} \frac{3}{5} y_1 (y_2 + y_1) dy_1 dy_2 = \int_1^2 \left[\frac{3 y_1^2 y_2}{10} + \frac{3 y_1^3}{15} \right] \bigg|_0^{1/2} dy_2$$

$$\int_1^2 \left(\frac{3 y_2}{40} + \frac{1}{40} \right) dy_2 = \left[\frac{3 y_2^2}{80} + \frac{y_2}{40} \right]_1^2 = 11/80$$

If Y_1, Y_2 are continuous r.v.s with joint distr. func. $F(y_1, y_2)$ if:
 exist a non negative func. $f(y_1, y_2)$ such that

$$F(y_1, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f(t_1, t_2) dt_1 dt_2 \quad \text{for } -\infty < y_1 < \infty, -\infty < y_2 < \infty$$

then y_1, y_2 are said to be jointly continuous r.v.s

If Y_1 & Y_2 are r.v.s with joint distr. func. $F(y_1, y_2)$

$$1 - F(-\infty, \infty) = F(-\infty, y_2) = F(y_1, -\infty) = 0$$

$$2 - F(\infty, \infty) = 1$$

3 - If $y_1^* \geq y_1$, $y_2^* \geq y_2$ then

$$F(y_1^*, y_2^*) - F(y_1^*, y_2) - F(y_1, y_2^*) + F(y_1, y_2) \geq 0$$

$$f(y_1, y_2) = \frac{\partial^2}{\partial y_1 \partial y_2} F(y_1, y_2)$$

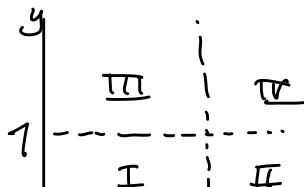
Joint distr. Func. of 2 r.v.s $\Rightarrow$ determine their joint pdf at all points (y_1, y_2) where joint density is continuous

Geometrically for two r.v. joint prob. is a surface, prob. is value under this surface

Ex: If joint pdf of X & Y is

$$f(x, y) = \begin{cases} x+y & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find joint distr. func. of these two random varbs.



If either $x < 0$, $y < 0$, $F(x, y) = 0$
 for $0 < x < 1$, $0 < y < 1$, Region I

Ex: find joint prob. density of two i.v.s x & y where joint distr. func. is

$$P(x, y) = \begin{cases} (1 - e^{-x})(1 - e^{-y}) & \text{for } x > 0, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Also use joint pdf. to determine $P(1 < x < 3, 1 < y < 2)$

$$\text{for } f(x, y) = \frac{\partial^2}{\partial x \partial y} P(x, y) = e^{-(x+y)}, \quad x > 0, y > 0$$

$$f(x, y) = \begin{cases} e^{-(x+y)} & , x > 0, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_1^2 \int_1^3 e^{-(x+y)} \cdot dx \cdot dy = (e^{-1} - e^{-3})(e^{-1} - e^{-2}) = e^{-2} - e^{-3} - e^{-4} + e^{-5} = 0.074$$

$$P(1 < x < 3, 1 < y < 2) = 0.074$$

Ex: If trivariable pdf of x_1, x_2, x_3 is

$$f(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2) e^{-x_3} & \text{for } 0 < x_1 < 1, 0 < x_2 < 1, x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find $P[(x_1, x_2, x_3) \in A]$ where A is a region

$$S(x_1, x_2, x_3) \mid 0 < x_1 < \frac{1}{2}, \frac{1}{2} < x_2 < 1, x_3 > 1 \}$$

$$P[(x_1, x_2, x_3) \in A] = \int_0^1 \int_{1/2}^1 \int_0^{1/2} (x_1 + x_2) e^{-x_3} \cdot dx_1 \cdot dx_2 \cdot dx_3$$

$$= \int_0^1 \int_{1/2}^1 \left(\frac{1}{8} + \frac{x_2}{2} \right) e^{-x_3} \cdot dx_2 \cdot dx_3 = \int_0^1 \frac{1}{4} \cdot e^{-x_3} \cdot dx_3$$

$$= 1(1 - e^{-1}) \approx 0.632$$

$$\frac{1}{4} (1 + 0.5) = 0.125$$