

2025-2026

ECON 205

Problem Set 5 Answers

1) Let the joint probability density function of X and Y be:

$$f(x, y) = \begin{cases} \frac{3}{8}y, & 0 \leq x \leq y, 0 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Compute the covariance between X and Y.

SOLUTION

1) We need to find marginal densities, E(X), E(Y) and E(XY).

$$f_x(x, y) = \int_{-\infty}^{\infty} f(x, y) dy = \int_x^2 \frac{3}{8}y dy = \frac{3}{16}y^2 \Big|_x^2 = \frac{3}{4} - \frac{3}{16}x^2 \quad \text{for } 0 \leq x \leq 2, \quad 0 \text{ otherwise}$$

$$E(X) = \int_{-\infty}^{\infty} x f_x(x, y) dx = \int_0^2 x \left(\frac{3}{4} - \frac{3}{16}x^2 \right) dx = \frac{3}{4} \int_0^2 x dx - \frac{3}{16} \int_0^2 x^3 dx = \frac{3}{4}$$

$$f_y(x, y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^y \frac{3}{8}y dx = \frac{3}{8}y \int_0^y dx = \frac{3}{8}y^2 \quad \text{for } 0 \leq y \leq 2, \quad 0 \text{ otherwise}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_y(x, y) dy = \int_0^2 y \frac{3}{8}y^2 dy = \frac{3}{2}$$

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x, y) dy dx = \int_0^2 \int_0^y xy \left(\frac{3}{8}y \right) dx dy = \frac{6}{5}$$

$$\text{Cov}(XY) = E(XY) - E(X)E(Y) = \frac{6}{5} - \frac{3}{4} \times \frac{3}{2} = \frac{3}{40}$$

2) Suppose that Y_1 and Y_2 are random variables and show that

$$V(Y_1 + Y_2) = V(Y_1) + V(Y_2) + 2\text{Cov}(Y_1, Y_2).$$

SOLUTION

2)

Proof. Let $Z = Y_1 + Y_2$. Using the definition of variance, we have

$$\begin{aligned} V(Z) &= E[(Z - \mu_Z)^2] \\ &= E\{[(Y_1 + Y_2) - E(Y_1 + Y_2)]^2\} \\ &= E[(Y_1 + Y_2 - \mu_{Y_1} - \mu_{Y_2})^2] \\ &= E\{[(Y_1 - \mu_{Y_1}) + (Y_2 - \mu_{Y_2})]^2\} \\ &= E[(Y_1 - \mu_{Y_1})^2 + (Y_2 - \mu_{Y_2})^2 + 2 \underbrace{(Y_1 - \mu_{Y_1})(Y_2 - \mu_{Y_2})}_{\text{cross product}}] \\ &= E[(Y_1 - \mu_{Y_1})^2] + E[(Y_2 - \mu_{Y_2})^2] + 2E[(Y_1 - \mu_{Y_1})(Y_2 - \mu_{Y_2})] \\ &= V(Y_1) + V(Y_2) + 2\text{Cov}(Y_1, Y_2). \end{aligned}$$

3) Let Y_1 and Y_2 have the joint probability function given by

$$f(y_1, y_2) = \begin{cases} ky_1 y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

- a) Find the value of k that makes this a probability density function.
- b) Find the cumulative distribution function for Y_1 and Y_2 .
- c) Find $P(Y_1 \leq \frac{1}{2}, Y_2 \leq \frac{3}{4})$

SOLUTION

a. Since the density must integrate to 1, evaluate $\int_0^1 \int_0^1 ky_1 y_2 dy_1 dy_2 = k/4 = 1$, so $k = 4$.

b. $F(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2) = 4 \int_0^{y_2} \int_0^{y_1} t_1 t_2 dt_1 dt_2 = y_1^2 y_2^2, 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1.$

$$F(y_1, y_2) = 0 \text{ if } y_1 < 0 \text{ or } y_2 < 0$$

$$F(y_1, y_2) = 1 \text{ if } y_1 > 1 \text{ and } y_2 > 1$$

$$F(y_1, y_2) = y_1^2 \text{ if } 0 \leq y_1 \leq 1 \text{ and } y_2 > 1$$

$$F(y_1, y_2) = y_2^2 \text{ if } y_1 > 1 \text{ and } 0 \leq y_2 \leq 1$$

c. $P(Y_1 \leq 1/2, Y_2 \leq 3/4) = (1/2)^2(3/4)^2 = 9/64.$

4) Find the marginal densities for Y_1 and Y_2 .

a) $f(y_1, y_2) = \begin{cases} 4y_1y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

b) $f(y_1, y_2) = \begin{cases} 6(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

SOLUTION

a)

$$f_1(y_1) = \int_0^1 4y_1y_2 dy_2 = 2y_1, \quad 0 \leq y_1 \leq 1; \quad f(y_2) = 2y_2, \quad 0 \leq y_2 \leq 1.$$

$$f_1(y_1) = \begin{cases} 2y_1, & 0 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad f_2(y_2) = \begin{cases} 2y_2, & 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

b)

$$f_1(y_1) = \int_{y_1}^1 6(1 - y_2) dy_2 = 3(1 - y_1)^2, \quad 0 \leq y_1 \leq 1;$$

$$f_2(y_2) = \int_0^{y_2} 6(1 - y_2) dy_1 = 6y_2(1 - y_2), \quad 0 \leq y_2 \leq 1.$$

$$f_1(y_1) = \begin{cases} 3(1 - y_1)^2, & 0 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad f_2(y_2) = \begin{cases} 6y_2(1 - y_2), & 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

5)

The random variable X has a range of $\{0,1,2\}$ and the random variable Y has a range of $\{1,2\}$. The joint distribution of X and Y is given by the following table.

		y	
		1	2
x	$p(x,y)$		
	0	0.2	0.1
	1	0.0	0.2
	2	0.3	0.2

- Write down tables for marginal distributions of X and Y .
- Compute $E(X)$ and $E(Y)$.
- Compute $E(XY)$

SOLUTION

- Write down tables for marginal distributions of X and Y .

To find marginal distributions of X and Y , we use the definitions below.

$$p_X(x) = \sum_y p(x,y)$$

$$p_Y(y) = \sum_x p(x,y)$$

Then we can write down the tables for marginal distributions of X and Y.

x	$p_X(x)$
0	0.3
1	0.2
2	0.5

y	$p_Y(y)$
1	0.5
2	0.5

- ii) Compute $E(X)$ and $E(Y)$.

$$E(X) = \sum_x x \cdot p_X(x) = 0 * (0.3) + 1 * (0.2) + 2 * (0.5) = 1.2$$

$$E(Y) = \sum_y y \cdot p_Y(y) = 1 * (0.5) + 2 * (0.5) = 1.5$$

- iii) Compute $E(XY)$

$$E(XY) = \sum_x \sum_y xy \cdot p(x, y)$$

$$E(XY) = (0 * 1)(0.2) + (0 * 2)(0.1) + (1 * 1)(0.0) + (1 * 2)(0.2) \\ + (2 * 1)(0.3) + (2 * 2)(0.2) = 1.8$$

6)

Let X and Y be jointly continuous random variables with joint pdf

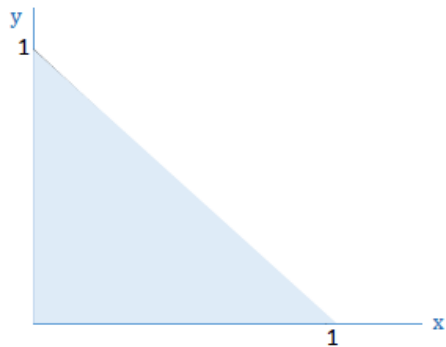
$$f(x, y) = \begin{cases} cx + 1, & x, y \geq 0, x + y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- Show the range of (X, Y) , in the x-y plane.
- Find the constant c .
- Find the marginal pdf's $f_X(x)$ and $f_Y(y)$.

- iv) Find $P(Y < 2X^2)$.

SOLUTION

- i) Show the range of (X, Y) , in the x-y plane.



- ii) Find the constant c .

To find the constant c , we write

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= 1 \\ \int_0^1 \int_0^{1-x} (cx + 1) dy dx &= 1 \\ \int_0^1 (cx + 1)(1 - x) dx &= 1 \\ \int_0^1 (-cx^2 + (c - 1)x + 1) dx &= 1 \\ \frac{1}{2} + \frac{1}{6}c &= 1 \rightarrow c = 3\end{aligned}$$

- iii) Find the marginal pdf's $f_X(x)$ and $f_Y(y)$.

We first note that $R_X = R_Y = [0, 1]$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

$$f_X(x) = \int_0^{1-x} (3x + 1) dy$$

$$f_X(x) = (3x + 1)(1 - x), \text{ for } x \in [0, 1]$$

Thus, we have

$$f_X(x) = \begin{cases} (3x + 1)(1 - x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we obtain

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

$$f_Y(y) = \int_0^{1-y} (3x + 1) dx$$

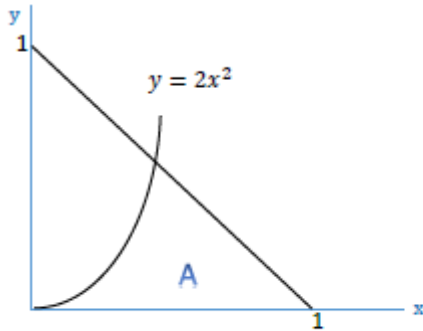
$$f_Y(y) = \frac{1}{2}(1 - y)(5 - 3y), \text{ for } y \in [0, 1]$$

Thus, we have

$$f_Y(y) = \begin{cases} \frac{1}{2}(1 - y)(5 - 3y), & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

iv) Find $P(Y < 2X^2)$.

To find $P(Y < 2X^2)$, we need to integrate $f(x, y)$ over the region A in figure below.



We have

$$P(Y < 2X^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{2x^2} f(x, y) dy dx$$

$$P(Y < 2X^2) = \int_0^1 \int_0^{\min(2x^2, 1-x)} (3x+1) dy dx$$

$$P(Y < 2X^2) = \int_0^1 (3x+1) \min(2x^2, 1-x) dx$$

$$P(Y < 2X^2) = \int_0^{\frac{1}{2}} 2x^2(3x+1) dx + \int_{\frac{1}{2}}^1 (3x+1)(1-x) dx$$

$$P(Y < 2X^2) = \left(\frac{3x^4}{2} + \frac{2x^3}{3} \right) \Big|_{x=0}^{x=1/2} + (-x^3 + x^2 + x) \Big|_{x=1/2}^{x=1}$$

$$P(Y < 2X^2) = \frac{53}{96}$$

7) Suppose you roll two fair, six-sided dice, one of which is red and the other of which is green. Define the following random variables:

X = The number shown on the red die

$Y = \begin{cases} 0 & \text{if the two dice show the same number} \\ 1 & \text{if the number on the green die is bigger than the number on the red die} \\ 2 & \text{if the number on the red die is bigger than the number on the green die} \end{cases}$

- Write down a table showing the joint probability mass function for X and Y .
- Find the marginal probability mass function for Y , and compute its expected value.

SOLUTION

a. Write down a table showing the joint probability mass function for X and Y.

		$X =$					
		1	2	3	4	5	6
$Y =$	0	1/36	1/36	1/36	1/36	1/36	1/36
	1	5/36	4/36	3/36	2/36	1/36	0
	2	0	1/36	2/36	3/36	4/36	5/36

b. Find the marginal probability mass function for Y, and compute its expected value.

y	$P(Y = y)$
0	6/36
1	15/36
2	15/36

$$E(Y) = 0 \times (6/36) + 1 \times (15/36) + 2 \times (15/36) = 45/36$$

8)

Assume we have the following form;

$$f(x, y) = \begin{cases} 6yx^2, & 0 < y < 1 \text{ and } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Is $f(x, y)$ a joint probability density function? If so calculate the following:

a) The marginal densities of X and Y.

b) $P(X + Y > 1)$

c) $P\left(\frac{1}{3} < Y < \frac{1}{2}\right)$.

SOLUTION

$$f(x, y) = \begin{cases} 6yx^2, & 0 < y < 1 \text{ and } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1 \quad \forall x, y \quad 0 \leq f(x, y)$$

$$\int_0^1 \int_0^1 6yx^2 dx dy = 1 \text{ since}$$

$$\int_0^1 6yx^2 dx = \frac{6yx^3}{3} \Big|_{x=0}^{x=1} = 2y$$

$$\int_0^1 2y dy = y^2 \Big|_{y=0}^{y=1} = 1$$

a) $f_x(x) = \int_{-\infty}^{\infty} f(x, y) dy$

If $0 < x < 1$, $f_x(x) = \int_0^1 6yx^2 dy = 3y^2 x^2 \Big|_{y=0}^{y=1} = 3x^2$

$$f_x(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

If $0 < y < 1$, $f_y(y) = \int_0^1 6yx^2 dx = 2yx^3 \Big|_{x=0}^{x=1} = 2y$

$$f_y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

b) $P(x+y > 1)$

$$\begin{aligned} P(y > 1-x) &= \int_0^1 \int_{1-x}^1 6yx^2 dy dx = \int_0^1 \left(3y^2 x^2 \right) \Big|_{y=1-x}^{y=1} dx = \int_0^1 (3x^2 - 3(1-x)^2 x^2) dx \\ &= \int_0^1 (6x^3 - 3x^4) dx = \frac{6}{4} x^4 \Big|_0^1 - \frac{3}{5} x^5 \Big|_0^1 = \frac{6}{4} - 0 - \left(\frac{3}{5} - 0 \right) = \frac{9}{10} \end{aligned}$$

c) $P\left(\frac{1}{3} < Y < \frac{1}{2}\right) = \int_{1/3}^{1/2} f_y(y) dy = \int_{1/3}^{1/2} 2y dy = y^2 \Big|_{1/3}^{1/2} = \frac{1}{4} - \frac{1}{9} = \frac{5}{36}$