

Econ 205

Multivariate (Joint) Probability Distributions – 2

Course Outline

- **Probability**

- Probability and Inference
- The Discrete Case
- Calculating the Probability of an Event
- Conditional Probability and the Independence of Events
- Laws of Probability
- The Law of Total Probability and Bayes' Rule
- Introduction to Random Variables and their distributions

- **Discrete Random Variables and Their Probability Distributions**

- Discrete Random Variable
- The Probability Distribution for Discrete Random Variable
- The Expected Value of Random Variable
- Moments and Moment-Generating Functions
- The Bernoulli Probability Distribution
- The Binomial Probability Distribution
- The Geometric Probability Distribution
- The Negative Binomial Probability Distribution
- The Hypergeometric Probability Distribution
- The Poisson Probability Distribution
- Probability-Generating Functions

- **Continuous Random Variables and Their Probability Densities**

- Continuous Random Variable
- The Probability Distribution for Continuous Random Variable
- The Expected Value for Continuous Random Variable
- Moments and Moment-Generating Functions
- The Uniform Probability Distribution
- The Normal Probability Distribution
- Exponential Probability Distribution
- Chi-Square Probability Distribution
- Tchebysheff's Theorem

Course Outline (cont'd)

- **Multivariate Probability Distributions**

- Bivariate and Multivariate Probability Distributions
- Independent Random Variables
- Marginal and Conditional Probability Distributions
- The Expected Value of a Function of Random Variables
- Special Theorems
- The Covariance of Two Random Variables
- The Expected Value and Variance of Linear Functions of Random Variables
- The Multinomial Probability Distribution
- The Bivariate Normal Distribution
- Conditional Expectations

- **Functions of Random Variables**

- Finding the Probability Distribution of a Function of Random Variables
- The Method of Distribution Functions
- The Method of Transformations
- The Method of Moment-Generating Functions
- Multivariable Transformations Using Jacobians

Table of Contents of Devore & Berk

2	Probability	50
	Introduction	50
2.1	Sample Spaces and Events	51
2.2	Axioms, Interpretations, and Properties of Probability	56
2.3	Counting Techniques	66
2.4	Conditional Probability	74
2.5	Independence	84
3	Discrete Random Variables and Probability Distributions	96
	Introduction	96
3.1	Random Variables	97
3.2	Probability Distributions for Discrete Random Variables	101
3.3	Expected Values of Discrete Random Variables	112
3.4	Moments and Moment Generating Functions	121
3.5	The Binomial Probability Distribution	128
3.6	Hypergeometric and Negative Binomial Distributions	138
3.7	The Poisson Probability Distribution	146
4	Continuous Random Variables and Probability Distributions	158
	Introduction	158
4.1	Probability Density Functions and Cumulative Distribution Functions	159
4.2	Expected Values and Moment Generating Functions	171
4.3	The Normal Distribution	179
4.4	The Gamma Distribution and Its Relatives	194
4.5	Other Continuous Distributions	202
4.6	Probability Plots	210
4.7	Transformations of a Random Variable	220
5	Joint Probability Distributions	232
	Introduction	232
5.1	Jointly Distributed Random Variables	233
5.2	Expected Values, Covariance, and Correlation	245
5.3	Conditional Distributions	253
5.4	Transformations of Random Variables	265
5.5	Order Statistics	271

Content of this presentation

- **Expected Values, Covariance, Correlation**
 - Expected Value of a function of random variables
 - Special Theorems
 - Covariance
 - Expected Value, Variance and Covariance of **Linear** Functions of Random Variables
 - Correlation Coefficient
- **Examples**

Expected Value of a function of random variables

Let X and Y be jointly distributed rv's with pmf $p(x, y)$ or pdf $f(x, y)$ according to whether the variables are discrete or continuous. Then the expected value of a function $h(X, Y)$, denoted by $E[h(X, Y)]$ or $\mu_{h(X,Y)}$ is given by

$$E[h(X, Y)] = \begin{cases} \sum_x \sum_y h(x, y) \cdot p(x, y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) \cdot f(x, y) dx dy & \text{if } X \text{ and } Y \text{ are continuous} \end{cases}$$

Compare the joint case with the earlier Univariate cases:

Discrete case:

If the rv X has a set of possible values D and pmf $p(x)$, then the expected value of any function $h(X)$, denoted by $E[h(X)]$ or $\mu_{h(X)}$, is computed by

$$E[h(X)] = \sum_D h(x) \cdot p(x)$$

assuming that $\sum_D |h(x)| \cdot p(x)$ is finite.

Continuous case:

If X is a continuous rv with pdf $f(x)$ and $h(X)$ is any function of X , then

$$E[h(X)] = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x) \cdot f(x) dx$$

This expected value will exist provided that $\int_{-\infty}^{\infty} |h(x)| f(x) dx < \infty$

Special theorems

THEOREM

Let c be a constant. Then

$$E(c) = c.$$

THEOREM

Let $g(Y_1, Y_2)$ be a function of the random variables Y_1 and Y_2 and let c be a constant. Then

$$E[cg(Y_1, Y_2)] = cE[g(Y_1, Y_2)].$$

THEOREM

Let Y_1 and Y_2 be random variables and $g_1(Y_1, Y_2), g_2(Y_1, Y_2), \dots, g_k(Y_1, Y_2)$ be functions of Y_1 and Y_2 . Then

$$\begin{aligned} E[g_1(Y_1, Y_2) + g_2(Y_1, Y_2) + \dots + g_k(Y_1, Y_2)] \\ = E[g_1(Y_1, Y_2)] + E[g_2(Y_1, Y_2)] + \dots + E[g_k(Y_1, Y_2)]. \end{aligned}$$

THEOREM

Let Y_1 and Y_2 be **independent** random variables and $g(Y_1)$ and $h(Y_2)$ be functions of only Y_1 and Y_2 , respectively. Then

$$E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)],$$

provided that the expectations exist.

Proof

We will give the proof of the result for the continuous case. Let $f(y_1, y_2)$ denote the joint density of Y_1 and Y_2 . The product $g(Y_1)h(Y_2)$ is a function of Y_1 and Y_2 . Hence, by Definition and the assumption that Y_1 and Y_2 are independent,

$$\begin{aligned} E[g(Y_1)h(Y_2)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y_1)h(y_2)f(y_1, y_2) dy_2 dy_1 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y_1)h(y_2)f_1(y_1)f_2(y_2) dy_2 dy_1 \\ &= \int_{-\infty}^{\infty} g(y_1)f_1(y_1) \left[\int_{-\infty}^{\infty} h(y_2)f_2(y_2) dy_2 \right] dy_1 \\ &= \int_{-\infty}^{\infty} g(y_1)f_1(y_1)E[h(Y_2)] dy_1 \\ &= E[h(Y_2)] \int_{-\infty}^{\infty} g(y_1)f_1(y_1) dy_1 = E[g(Y_1)]E[h(Y_2)]. \end{aligned}$$

The proof for the discrete case follows in an analogous manner.

PROPOSITION

Let $X_1, X_2, \dots, X_n$ be **independent** random variables and assume that the expected values of $h_1(X_1), h_2(X_2), \dots, h_n(X_n)$ all exist. Then

$$E[h_1(X_1) \cdot h_2(X_2) \cdot \dots \cdot h_n(X_n)] = E[h_1(X_1)] \cdot E[h_2(X_2)] \cdot \dots \cdot E[h_n(X_n)]$$

Covariance

Definition:

The **covariance** between two rv's X and Y is

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

$$= \begin{cases} \sum_x \sum_y (x - \mu_X)(y - \mu_Y)p(x, y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y)f(x, y) dx dy & \text{if } X \text{ and } Y \text{ are continuous} \end{cases}$$

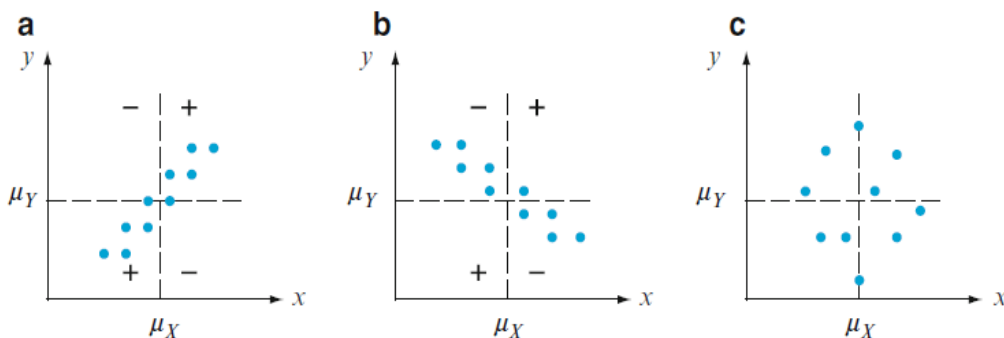


Figure 1.1. $p(x, y) = \frac{1}{10}$ for each of ten pairs corresponding to indicated points; (a) positive covariance; (b) negative covariance; (c) covariance near zero

THEOREM

If Y_1 and Y_2 are random variables with means μ_1 and μ_2 , respectively, then

$$\text{Cov}(Y_1, Y_2) = E[(Y_1 - \mu_1)(Y_2 - \mu_2)] = E(Y_1 Y_2) - E(Y_1)E(Y_2).$$

Proof

$$\begin{aligned} \text{Cov}(Y_1, Y_2) &= E[(Y_1 - \mu_1)(Y_2 - \mu_2)] \\ &= E(Y_1 Y_2 - \mu_1 Y_2 - \mu_2 Y_1 + \mu_1 \mu_2). \end{aligned}$$

From Theorem 1.1, the expected value of a sum is equal to the sum of the expected values; and from Theorem 1.2, the expected value of a constant times a function of random variables is the constant times the expected value. Thus,

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - \mu_1 E(Y_2) - \mu_2 E(Y_1) + \mu_1 \mu_2.$$

Because $E(Y_1) = \mu_1$ and $E(Y_2) = \mu_2$, it follows that

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = E(Y_1 Y_2) - \mu_1 \mu_2.$$

Corollary: $\text{Cov}(X, X) = E(X^2) - \mu_X^2 = V(X).$

Covariance (cont'd)

THEOREM

If Y_1 and Y_2 are independent random variables, then

$$\text{Cov}(Y_1, Y_2) = 0.$$

Thus, independent random variables must be uncorrelated.

Proof

Theorem 10.1 establishes that

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - \mu_1 \mu_2.$$

Because Y_1 and Y_2 are independent, Theorem 10.1 implies that

$$E(Y_1 Y_2) = E(Y_1)E(Y_2) = \mu_1 \mu_2,$$

and the desired result follows immediately.

The converse of this theorem is not true in general!

Expected Value, Variance and Covariance of Linear Functions of Random Variables

PROPOSITION If X, Y , and Z are rv's and a and b are constants then

$$\text{Cov}(aX + bY, Z) = a \text{Cov}(X, Z) + b \text{Cov}(Y, Z)$$

The proposition above is a special case of part c of the the following theorem

THEOREM

Let $Y_1, Y_2, \dots, Y_n$ and $X_1, X_2, \dots, X_m$ be random variables with $E(Y_i) = \mu_i$ and $E(X_j) = \xi_j$. Define

$$U_1 = \sum_{i=1}^n a_i Y_i \quad \text{and} \quad U_2 = \sum_{j=1}^m b_j X_j$$

for constants $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$. Then the following hold:

- a** $E(U_1) = \sum_{i=1}^n a_i \mu_i$.
- b** $V(U_1) = \sum_{i=1}^n a_i^2 V(Y_i) + 2 \sum \sum_{1 \leq i < j \leq n} a_i a_j \text{Cov}(Y_i, Y_j)$, where the double sum is over all pairs (i, j) with $i < j$.
- c** $\text{Cov}(U_1, U_2) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \text{Cov}(Y_i, X_j)$.

Proof

The theorem consists of three parts, of which (a) follows directly from Theorems . To prove (b), we appeal to the definition of variance and write

$$\begin{aligned} V(U_1) &= E[U_1 - E(U_1)]^2 = E\left[\sum_{i=1}^n a_i Y_i - \sum_{i=1}^n a_i \mu_i\right]^2 \\ &= E\left[\sum_{i=1}^n a_i (Y_i - \mu_i)\right]^2 \\ &= E\left[\sum_{i=1}^n a_i^2 (Y_i - \mu_i)^2 + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n a_i a_j (Y_i - \mu_i)(Y_j - \mu_j)\right] \\ &= \sum_{i=1}^n a_i^2 E(Y_i - \mu_i)^2 + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n a_i a_j E[(Y_i - \mu_i)(Y_j - \mu_j)]. \end{aligned}$$

By the definitions of variance and covariance, we have

$$V(U_1) = \sum_{i=1}^n a_i^2 V(Y_i) + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n a_i a_j \text{Cov}(Y_i, Y_j).$$

Because $\text{Cov}(Y_i, Y_j) = \text{Cov}(Y_j, Y_i)$, we can write

$$V(U_1) = \sum_{i=1}^n a_i^2 V(Y_i) + 2 \sum \sum_{1 \leq i < j \leq n} a_i a_j \text{Cov}(Y_i, Y_j).$$

Similar steps can be used to obtain (c). We have

$$\begin{aligned} \text{Cov}(U_1, U_2) &= E\{[U_1 - E(U_1)][U_2 - E(U_2)]\} \\ &= E\left[\left(\sum_{i=1}^n a_i Y_i - \sum_{i=1}^n a_i \mu_i\right)\left(\sum_{j=1}^m b_j X_j - \sum_{j=1}^m b_j \xi_j\right)\right] \\ &= E\left\{\left[\sum_{i=1}^n a_i (Y_i - \mu_i)\right]\left[\sum_{j=1}^m b_j (X_j - \xi_j)\right]\right\} \end{aligned}$$

$$\begin{aligned} &= E\left[\sum_{i=1}^n \sum_{j=1}^m a_i b_j (Y_i - \mu_i)(X_j - \xi_j)\right] \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j E[(Y_i - \mu_i)(X_j - \xi_j)] \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \text{Cov}(Y_i, X_j). \end{aligned}$$

On observing that $\text{Cov}(Y_i, Y_i) = V(Y_i)$, we can see that (b) is a special case of (c).

Correlation Coefficient

DEFINITION The **correlation coefficient** of X and Y , denoted by $\text{Corr}(X, Y)$, or $\rho_{X,Y}$, or just ρ , is defined by

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sigma_X \cdot \sigma_Y}$$

PROPOSITION 1. If a and c are either both positive or both negative,

$$\text{Corr}(aX + b, cY + d) = \text{Corr}(X, Y)$$

2. For any two rv's X and Y , $-1 \leq \text{Corr}(X, Y) \leq 1$

PROPOSITION 1. If X and Y are independent, then $\rho = 0$, but $\rho = 0$ does not imply independence.
2. $\rho = 1$ or -1 iff $Y = aX + b$ for some numbers a and b with $a \neq 0$.

EXAMPLES

Example 1: Expected Value of a function of random variables - Discrete Case

Example

Five friends have purchased tickets to a concert. If the tickets are for seats 1–5 in a particular row and the tickets are randomly distributed among the five, what is the expected number of seats separating any particular two of the five? Let X and Y denote the seat numbers of the first and second individuals, respectively. Possible (X, Y) pairs are $\{(1, 2), (1, 3), \dots, (5, 4)\}$, and the joint pmf of (X, Y) is

$$p(x, y) = \begin{cases} \frac{1}{20} & x = 1, \dots, 5; \quad y = 1, \dots, 5; \quad x \neq y \\ 0 & \text{otherwise} \end{cases}$$

The number of seats separating the two individuals is $h(X, Y) = |X - Y| - 1$. The accompanying table gives $h(x, y)$ for each possible (x, y) pair.

$h(x, y)$		x				
		1	2	3	4	5
y	1	–	0	1	2	3
	2	0	–	0	1	2
	3	1	0	–	0	1
	4	2	1	0	–	0
	5	3	2	1	0	–

Thus

$$E[h(X, Y)] = \sum_{(x, y)} \sum h(x, y) \cdot p(x, y) = \sum_{\substack{x=1 \\ x \neq y}}^5 \sum_{y=1}^5 (|x - y| - 1) \cdot \frac{1}{20} = 1 \quad \blacksquare$$

Definition of $E[h(X, Y)]$:

Let X and Y be jointly distributed rv's with pmf $p(x, y)$ or pdf $f(x, y)$ according to whether the variables are discrete or continuous. Then the expected value of a function $h(X, Y)$, denoted by $E[h(X, Y)]$ or $\mu_{h(X, Y)}$ is given by

$$E[h(X, Y)] = \begin{cases} \sum_x \sum_y h(x, y) \cdot p(x, y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) \cdot f(x, y) dx dy & \text{if } X \text{ and } Y \text{ are continuous} \end{cases}$$

Example 2: Expected Value of a function of random variables - Continuous Case

EXAMPLE Let Y_1 and Y_2 have joint density given by

$$f(y_1, y_2) = \begin{cases} 2y_1, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $E(Y_1 Y_2)$.

Solution From Definition we obtain

$$\begin{aligned} E(Y_1 Y_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1 y_2 f(y_1, y_2) dy_1 dy_2 = \int_0^1 \int_0^1 y_1 y_2 (2y_1) dy_1 dy_2 \\ &= \int_0^1 y_2 \left(\frac{2y_1^3}{3} \Big|_0^1 \right) dy_2 = \int_0^1 \left(\frac{2}{3} \right) y_2 dy_2 = \frac{2}{3} \frac{y_2^2}{2} \Big|_0^1 = \frac{1}{3}. \end{aligned}$$

Definition of $E[h(X,Y)]$:

Let X and Y be jointly distributed rv's with pmf $p(x, y)$ or pdf $f(x, y)$ according to whether the variables are discrete or continuous. Then the expected value of a function $h(X, Y)$, denoted by $E[h(X, Y)]$ or $\mu_{h(X,Y)}$ is given by

$$E[h(X, Y)] = \begin{cases} \sum_x \sum_y h(x, y) \cdot p(x, y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) \cdot f(x, y) dx dy & \text{if } X \text{ and } Y \text{ are continuous} \end{cases}$$

Example 3: Covariance – Continuous Case

EXAMPLE Let Y_1 and Y_2 have joint density given by

$$f(y_1, y_2) = \begin{cases} 2y_1, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the covariance of Y_1 and Y_2 .

Recall:

THEOREM

If Y_1 and Y_2 are random variables with means μ_1 and μ_2 , respectively, then

$$\text{Cov}(Y_1, Y_2) = E[(Y_1 - \mu_1)(Y_2 - \mu_2)] = E(Y_1 Y_2) - E(Y_1)E(Y_2).$$

Solution:

$$E(Y_1) = \int_0^1 \int_0^1 y_1(2y_1) dy_1 dy_2$$

$$= \int_0^1 \left(\frac{2y_1^3}{3} \right) \Big|_0^1 dy_2 = \int_0^1 \frac{2}{3} dy_2 = \frac{2}{3} y_2 \Big|_0^1 = \frac{2}{3}.$$

$$E(Y_2) = \int_0^1 \int_0^1 y_2(2y_1) dy_1 dy_2 = \int_0^1 y_2 \left(\frac{2y_1^2}{2} \right) \Big|_0^1 dy_2$$

$$= \int_0^1 y_2 dy_2 = \frac{y_2^2}{2} \Big|_0^1 = \frac{1}{2}.$$

Note that this joint density function is the same as the one in Example 2. So,

From Example 2, $E(Y_1 Y_2) = 1/3$. Also, $E(Y_1) = 2/3$ and $\mu_2 = E(Y_2) = 1/2$, so

$$\mu_1 =$$

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - \mu_1 \mu_2 = (1/3) - (2/3)(1/2) = 0. \quad \blacksquare$$

Example 4: Covariance – Discrete Case

EXAMPLE Let Y_1 and Y_2 be discrete random variables with joint probability distribution as shown in Table 5.3. Show that Y_1 and Y_2 are dependent but have zero covariance.

Table 5.3 Joint probability distribution

y_2	y_1		
	-1	0	+1
-1	1/16	3/16	1/16
0	3/16	0	3/16
+1	1/16	3/16	1/16

Solution Calculation of marginal probabilities yields $p_1(-1) = p_1(1) = 5/16 = p_2(-1) = p_2(1)$, and $p_1(0) = 6/16 = p_2(0)$. The value $p(0, 0) = 0$ in the center cell stands out. Obviously,

$$p(0, 0) \neq p_1(0)p_2(0),$$

and this is sufficient to show that Y_1 and Y_2 are dependent.

Again looking at the marginal probabilities, we see that $E(Y_1) = E(Y_2) = 0$. Also,

$$\begin{aligned}
 E(Y_1 Y_2) &= \sum_{\text{all } y_1} \sum_{\text{all } y_2} y_1 y_2 p(y_1, y_2) \\
 &= (-1)(-1)(1/16) + (-1)(0)(3/16) + (-1)(1)(1/16) \\
 &\quad + (0)(-1)(3/16) + (0)(0)(0) + (0)(1)(3/16) \\
 &\quad + (1)(-1)(1/16) + (1)(0)(3/16) + (1)(1)(1/16) \\
 &= (1/16) - (1/16) - (1/16) + (1/16) = 0.
 \end{aligned}$$

Thus,

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = 0 - 0(0) = 0.$$

If the covariance of two random variables is zero, the variables *need not* be independent. ■

Recall:

THEOREM

If Y_1 and Y_2 are random variables with means μ_1 and μ_2 , respectively, then

$$\text{Cov}(Y_1, Y_2) = E[(Y_1 - \mu_1)(Y_2 - \mu_2)] = E(Y_1 Y_2) - E(Y_1)E(Y_2).$$

This example shows that the converse of the following theorem is not true:

THEOREM

If Y_1 and Y_2 are independent random variables, then

$$\text{Cov}(Y_1, Y_2) = 0.$$

Thus, independent random variables must be uncorrelated.

Example 5: Expected Value, Variance and Covariance of Linear Functions of Random Variables

EXAMPLE Let Y_1, Y_2 , and Y_3 be random variables, where $E(Y_1) = 1$, $E(Y_2) = 2$, $E(Y_3) = -1$, $V(Y_1) = 1$, $V(Y_2) = 3$, $V(Y_3) = 5$, $\text{Cov}(Y_1, Y_2) = -0.4$, $\text{Cov}(Y_1, Y_3) = 1/2$, and $\text{Cov}(Y_2, Y_3) = 2$. Find the expected value and variance of $U = Y_1 - 2Y_2 + Y_3$. If $W = 3Y_1 + Y_2$, find $\text{Cov}(U, W)$.

Solution $U = a_1Y_1 + a_2Y_2 + a_3Y_3$, where $a_1 = 1$, $a_2 = -2$, and $a_3 = 1$. Then by Theorem

$$E(U) = a_1E(Y_1) + a_2E(Y_2) + a_3E(Y_3) = (1)(1) + (-2)(2) + (1)(-1) = -4.$$

Similarly,

$$\begin{aligned} V(U) &= a_1^2V(Y_1) + a_2^2V(Y_2) + a_3^2V(Y_3) + 2a_1a_2\text{Cov}(Y_1, Y_2) \\ &\quad + 2a_1a_3\text{Cov}(Y_1, Y_3) + 2a_2a_3\text{Cov}(Y_2, Y_3) \\ &= (1)^2(1) + (-2)^2(3) + (1)^2(5) + (2)(1)(-2)(-0.4) \\ &\quad + (2)(1)(1)(1/2) + (2)(-2)(1)(2) \\ &= 12.6. \end{aligned}$$

Notice that $W = b_1Y_1 + b_2Y_2$, where $b_1 = 3$ and $b_2 = 1$. Thus,

$$\begin{aligned} \text{Cov}(U, W) &= a_1b_1\text{Cov}(Y_1, Y_1) + a_1b_2\text{Cov}(Y_1, Y_2) + a_2b_1\text{Cov}(Y_2, Y_1) \\ &\quad + a_2b_2\text{Cov}(Y_2, Y_2) + a_3b_1\text{Cov}(Y_3, Y_1) + a_3b_2\text{Cov}(Y_3, Y_2). \end{aligned}$$

Notice that $\text{Cov}(Y_i, Y_j) = \text{Cov}(Y_j, Y_i)$ and $\text{Cov}(Y_i, Y_i) = V(Y_i)$. Therefore,

$$\begin{aligned} \text{Cov}(U, W) &= (1)(3)(1) + (1)(1)(-0.4) + (-2)(3)(-0.4) \\ &\quad + (-2)(1)(3) + (1)(3)(1/2) + (1)(1)(2) \\ &= 2.5. \end{aligned}$$

Because $\text{Cov}(U, W) \neq 0$, it follows that U and W are *dependent*. ■

Recall:

THEOREM

Let $Y_1, Y_2, \dots, Y_n$ and $X_1, X_2, \dots, X_m$ be random variables with $E(Y_i) = \mu_i$ and $E(X_j) = \xi_j$. Define

$$U_1 = \sum_{i=1}^n a_i Y_i \quad \text{and} \quad U_2 = \sum_{j=1}^m b_j X_j$$

for constants $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$. Then the following hold:

- a $E(U_1) = \sum_{i=1}^n a_i \mu_i$.
- b $V(U_1) = \sum_{i=1}^n a_i^2 V(Y_i) + 2 \sum_{1 \leq i < j \leq n} a_i a_j \text{Cov}(Y_i, Y_j)$, where the double sum is over all pairs (i, j) with $i < j$.
- c $\text{Cov}(U_1, U_2) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \text{Cov}(Y_i, X_j)$.

Example 6: Expected Value, Variance and Covariance of Linear Functions of Random Variables

Recall:

THEOREM

Let $Y_1, Y_2, \dots, Y_n$ and $X_1, X_2, \dots, X_m$ be random variables with $E(Y_i) = \mu_i$ and $E(X_j) = \xi_j$. Define

$$U_1 = \sum_{i=1}^n a_i Y_i \quad \text{and} \quad U_2 = \sum_{j=1}^m b_j X_j$$

for constants $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$. Then the following hold:

- a $E(U_1) = \sum_{i=1}^n a_i \mu_i$.
- b $V(U_1) = \sum_{i=1}^n a_i^2 V(Y_i) + 2 \sum \sum_{1 \leq i < j \leq n} a_i a_j \text{Cov}(Y_i, Y_j)$, where the double sum is over all pairs (i, j) with $i < j$.
- c $\text{Cov}(U_1, U_2) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \text{Cov}(Y_i, X_j)$.

Example: Suppose that the joint density function

for Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the variance of

$$Y_1 - Y_2.$$

Solution Using Theorem , we have

$$V(Y_1 - Y_2) = V(Y_1) + V(Y_2) - 2 \text{Cov}(Y_1, Y_2).$$

Because

$$f_1(y_1) = \begin{cases} 3y_1^2, & 0 \leq y_1 \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$f_2(y_2) = \begin{cases} (3/2)(1 - y_2^2), & 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

it follows that

$$E(Y_1^2) = \int_0^1 3y_1^4 dy_1 = \frac{3}{5},$$

$$E(Y_2^2) = \int_0^1 \frac{3}{2} y_2^2 (1 - y_2^2) dy_2 = \frac{3}{2} \left[\frac{1}{3} - \frac{1}{5} \right] = \frac{1}{5}.$$

we have $E(Y_1) = 3/4$ and $E(Y_2) = 3/8$. Thus,

$$V(Y_1) = (3/5) - (3/4)^2 = .04 \quad \text{and} \quad V(Y_2) = (1/5) - (3/8)^2 = .06.$$

Example 6 (Cont'd):

$$\begin{aligned} E(Y_1 Y_2) &= \int_0^1 \int_0^{y_1} y_1 y_2 (3y_1) dy_2 dy_1 = \int_0^1 3y_1^2 \left(\frac{y_2^2}{2} \right) \Big|_0^{y_1} dy_1 \\ &= \int_0^1 \frac{3}{2} y_1^4 dy_1 = \frac{3}{2} \left(\frac{y_1^5}{5} \right) \Big|_0^1 = \frac{3}{10}. \end{aligned}$$

$E(Y_1) = 3/4$ and $E(Y_2) = 3/8$. Thus, using

Theorem 1.10, we obtain

$$\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = (3/10) - (3/4)(3/8) = .30 - .28 = .02.$$

we determined that $\text{Cov}(Y_1, Y_2) = .02$. Therefore,

$$\begin{aligned} V(Y_1 - Y_2) &= V(Y_1) + V(Y_2) - 2 \text{Cov}(Y_1, Y_2) \\ &= .04 + .06 - 2(.02) = .06. \end{aligned}$$

The standard deviation of $Y_1 - Y_2$ is then $\sqrt{.06} = .245$. ■

Recall:

THEOREM

Let $Y_1, Y_2, \dots, Y_n$ and $X_1, X_2, \dots, X_m$ be random variables with $E(Y_i) = \mu_i$ and $E(X_j) = \xi_j$. Define

$$U_1 = \sum_{i=1}^n a_i Y_i \quad \text{and} \quad U_2 = \sum_{j=1}^m b_j X_j$$

for constants $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$. Then the following hold:

- a $E(U_1) = \sum_{i=1}^n a_i \mu_i$.
- b $V(U_1) = \sum_{i=1}^n a_i^2 V(Y_i) + 2 \sum \sum_{1 \leq i < j \leq n} a_i a_j \text{Cov}(Y_i, Y_j)$, where the double sum is over all pairs (i, j) with $i < j$.
- c $\text{Cov}(U_1, U_2) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \text{Cov}(Y_i, X_j)$.

Example 7: Correlation Coefficient

Example

Let X and Y be discrete rv's with joint pmf

$$p(x, y) = \begin{cases} \frac{1}{4} & (x, y) = (-4, 1), (4, -1), (2, 2), (-2, -2) \\ 0 & \text{otherwise} \end{cases}$$

The points that receive positive probability mass are identified on the (x, y) coordinate system in Figure 5.5. It is evident from the figure that the value of X is completely determined by the value of Y and vice versa, so the two variables are completely dependent. However, by symmetry $\mu_X = \mu_Y = 0$ and $E(XY) = (-4)\frac{1}{4} + (-4)\frac{1}{4} + (4)\frac{1}{4} + (4)\frac{1}{4} = 0$ so $\text{Cov}(X, Y) = E(XY) - \mu_X \cdot \mu_Y = 0$ and thus $\rho_{X,Y} = 0$. Although there is perfect dependence, there is also complete absence of any linear relationship!

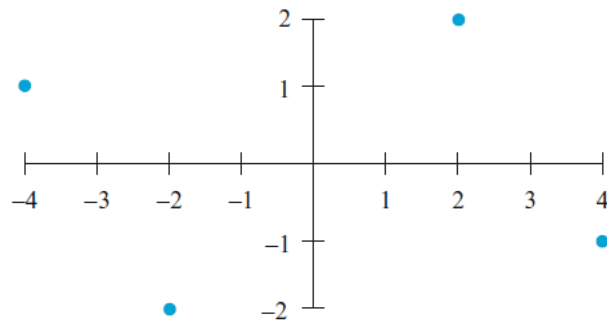


Figure 5.5 The population of pairs for Example

Recall:

DEFINITION

The **correlation coefficient** of X and Y , denoted by $\text{Corr}(X, Y)$, or $\rho_{X,Y}$, or just ρ , is defined by

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sigma_X \cdot \sigma_Y}$$

References

- Devore, J.L. and Berk, K.N. (2012) Modern Mathematical Statistics with Applications, 2nd Edition, New York: Springer.
- Wackerly, D.D., Mendenhall, W. and Scheaffer, R.L., (2008) Mathematical Statistics With Applications, 7th Edition, Pacific Grove: Duxbury.